A CHARACTERISATION OF LIPSCHITZ CLASSES ON FINITE DIMENSIONAL GROUPS

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Abstract. An analogue of a theorem of S. N. Bernstein is developed for certain metric locally compact abelian groups. This, together with a corresponding Jackson-type theorem, gives a characterisation in terms of their Fourier transforms of the Lipschitz functions defined on a compact abelian group with finite topological dimension.

Let $G$ denote a metric locally compact abelian (LCA) group, with translation-invariant metric $d$, and character group $\Gamma$. We shall choose Haar measures $\lambda, \theta$ for $G, \Gamma$ respectively so that Plancherel's theorem is valid.

It will be necessary to specify metrics for various standard groups, together with their finite products and homomorphic images. The real line $\mathbb{R}$ will be taken with its usual Euclidean metric. For any infinite first countable 0-dimensional LCA group $G$ we take a neighbourhood basis $(V'_n)$ at zero consisting of a strictly decreasing sequence of compact open subgroups of $G$ (for the existence of such a basis see [4, (7.7)]), any strictly decreasing sequence $(\beta_n)$ of positive numbers tending to zero, and define $d$ on $G \times G$ by

$$d(x,y) = \begin{cases} \beta_{n+1}, & x - y \in V'_n \setminus V'_{n+1}, \\ \beta_1, & x - y \not\in V'_1, \\ 0, & x = y. \end{cases}$$

It is easy to verify that $d$ so obtained is a translation-invariant metric on $G$ which generates the given topology. In the particular case when $G = \Delta_a$, the group of $a$-adic integers, where $a = (a_0, a_1, \ldots)$ and each $a_n$ is an integer greater than 1, we take the previous metric defined with respect to the basis $(\Lambda_n)$; here $\Lambda_n$ is the compact open subgroup of $\Delta_a$ given by

$$\Lambda_n = \{x \in \Delta_a : x_k = 0 \text{ for } k < n\}.$$ 

Given metric LCA groups $(G, d), (G', d')$, the product group $G \times G'$ will always be metrised by

$$d''((x, x'), (y, y')) = \max\{d(x, y), d'(x', y')\}.$$ 

If $H$ is a closed subgroup of $G$ we metrise the quotient group $G/H$ by
\[ d^*(x + H, y + H) = \inf\{d(a, b): a \in x + H, b \in y + H\}. \]

The character group \( \Gamma_{G \times G'} \) of a product \( G \times G' \) is topologically isomorphic with the product \( \Gamma_G \times \Gamma_{G'} \); a typical element \([\gamma, \gamma']\) of \( \Gamma_{G \times G'} \) is defined by
\[ [\gamma, \gamma']((x, x')) = \gamma(x)\gamma'(x'), \quad (x, x') \in G \times G', \]
where \( \gamma \in \Gamma_G, \gamma' \in \Gamma_{G'} \). Given sets \( T \subseteq \Gamma_G, T' \subseteq \Gamma_{G'} \) we shall write
\[ [T, T'] = \{[\gamma, \gamma']: \gamma \in T, \gamma' \in T'\}. \]

If \( H \) is a closed subgroup of \( G \) then the character group \( \Gamma_{G/H} \) of \( G/H \) is topologically isomorphic with \( A(\Gamma, H) \) (the annihilator of \( H \) in \( \Gamma \)) where, to each \( \gamma \in A(\Gamma, H) \), there corresponds \( \gamma^+ \in \Gamma_{G/H} \) such that
\[ \gamma^+(x + H) = \gamma(x), \quad x \in G. \]

Given \( \Xi \subseteq A(\Gamma, H) \) we write
\[ \Xi^+ = \{\gamma^+ \in \Gamma_{G/H}: \gamma \in \Xi\}. \]

We shall also denote by \( \tau_H \) the natural homomorphism of \( G \) onto \( G/H \).

The theorems of Jackson and Bernstein (see [6, Chapter 3, Theorems (13.6), (13.20)]) respectively) connect the modulus of continuity of a function \( f \) with the degree of approximation of \( f \) by functions with (certain) compact spectra. The mean modulus of continuity with exponent \( p \) of \( f \) is given by
\[ \omega(p; f; \delta) = \sup\{||f - f'||_p: d(a, 0) \leq \delta\}, \]
where \( \tau_a f: x \rightarrow f(x - a) \). If \( f \in L^p(G) \) has the property
\[ \omega(p; f; \delta) = O(\delta^\alpha), \quad \delta > 0, \]
for some \( \alpha > 0 \), then we say that \( f \) is of Lipschitz order \( \alpha \). The functions of Lipschitz order \( \alpha \) form a subspace of \( L^p(G) \), which we denote by \( \text{Lip}_p^\alpha \).

The spectrum \( \Sigma(f) \) is defined as in [4, (40.21)] for \( f \in L^\infty(G) \), and by
\[ \Sigma(f) = \bigcup \{\Sigma(\phi \ast f): \phi \in C_{00}(G)\} \]
(where \( C_{00}(G) \) denotes the space of continuous functions on \( G \) with compact support) for \( f \in L^p(G) \), \( p \in [1, \infty) \). When \( p = 1 \) we find that \( \Sigma(f) = \text{supp}(\hat{f}) \).

We write
\[ L^p_T(G) = \{f \in L^p(G): \Sigma(f) \subseteq T\}, \]
\[ E_T(p; f) = \inf\{||f - l||_p: l \in L^p_T(G)\}, \]
\[ \omega_T(a) = \sup\{|\gamma(a) - 1|: \gamma \in T\}, \quad a \in G; \]
in the above expressions \( T \) is a nonvoid subset of \( \Gamma \).

Our first result is the following analogue of Bernstein's theorem:

**Theorem 1.** Let \( G \) be a metric LCA group. Suppose we have an ascending family \( \{T_n\}_{n=1}^\infty \) of symmetric compact neighbourhoods of zero in \( \Gamma \), a sequence
(\beta_n) of positive numbers smaller than 1, and positive constants \(C, K, \mu (\mu < 1)\) such that for each \(n \in \{1, 2, \ldots\}\), (a) \(\omega_{\tau_n}(a) \leq C\beta_n^{-1}d(a, 0), a \in G\), (b) \(\beta_{n+1} \leq \mu \beta_n\), (c) \(\theta(3T_n) \leq K\theta(T_n)\).

Then any \(f \in L^p(G)\) with the property that \(\mathcal{E}_{\tau_n}(p; f) = O(\beta_{n+1}^a)\) for some \(a > 0\) satisfies

\[
\omega(p; f; \delta) = O(\delta^a), \quad 0 < a < 1,
\]
\[
= O(\delta|\log \delta|), \quad a = 1,
\]
\[
= O(\delta), \quad a > 1.
\]

**Proof.** Firstly we note that, with a slight modification of the proof, the lemma in [1] is valid for any LCA group (consider weak*-cluster points rather than weak*-convergent subsequences). This guarantees that for each \(n\) there exists \(t_n^* \in L^G(G)\) for which \(\mathcal{E}_{\tau_n}(p; f) = \|f - t_n^*\|_p\). By assumption

\[
\|f - t_n^*\|_p \leq B\beta_{n+1}^a
\]

for some constant \(B > 0\). Defining

\[
s_1 = t_1^*; \quad s_n - t_n^* - t_{n-1}^* \quad (n \in \{2, 3, \ldots\})
\]

we have (recall that \((\beta_n)\) is decreasing)

\[
\|s_n\|_p \leq \|s_n - f\|_p + \|f - t_{n-1}^*\|_p \leq 2B\beta_{n+1}^a \quad (n \in \{2, 3, \ldots\}).
\]

Hence we can find \(B' > 0\) such that for all \(n \in \{1, 2, \ldots\}\)

\[
(1) \quad \|s_n\|_p \leq B'\beta_n^a.
\]

Now \(\sum_{k=1}^n s_k = t_n^*\) converges in \(L^p(G)\) to \(f\) as \(n \to \infty\). Consequently, for any \(a \in G\), \(\tau_a(\sum_{k=1}^n s_k) - \sum_{k=1}^n s_k\) converges in \(L^p(G)\) to \(\tau_a f - f\) as \(n \to \infty\), and

\[
(2) \quad \|\tau_a f - f\|_p \leq \sum_{k=1}^\infty \|\tau_a s_k - s_k\|_p \leq \sum_{k=1}^m \|\tau_a s_k - s_k\|_p + 2 \sum_{k=m+1}^\infty \|s_k\|_p.
\]

The proof of [2, Theorem 1.3] can be adapted to show that

\[
\|\tau_a s_k - s_k\|_p \leq 4\left(\theta(3\tau_k)/\theta(\tau_k)\right)^{1/2} \omega_{\tau_k}(a)\|s_k\|_p
\]
\[
\leq 4K^{1/2} C\beta_k^{-1}d(a, 0)\|s_k\|_p,
\]
the last step using (a) and (c) above. A combination of (1), (2) and (3) gives

\[
\omega(p; f; \delta) \leq 4K^{1/2} CB\delta \sum_{k=1}^m \beta_k^{a-1} + 2B' \sum_{k=m+1}^\infty \beta_k^a
\]

for any \(\delta > 0\).
Now suppose that \( 0 < \delta \leq \beta_1 \), and choose \( m \geq 1 \) so that \( \beta_{m+1} < \delta \leq \beta_m \). Then, using (b),

\[
\omega(p; f; \delta) \leq 4K^{1/2}CB'\delta \sum_{k=1}^{m} \beta_k^{a-1} + 2B'\delta^a \sum_{k=m+1}^{\infty} \left( \frac{\beta_k}{\beta_{m+1}} \right)^a
\]

\[
\leq 4K^{1/2}CB'\delta \sum_{k=1}^{m} \beta_k^{a-1} + \frac{2B'}{1 - \mu^a} \delta^a.
\]

The estimates in the statement of the theorem now follow easily as in the proof of \([1, \text{Theorem 1}]\).

We shall determine some metric groups for which families \( \{T_n\}_{n=1}^{\infty}, (\beta_n) \) can be found satisfying (a)-(c) above.

The classical examples are the real line \( \mathbb{R} \) and the circle group \( \mathbb{T} \). For \( G = \mathbb{R} \), just put \( T_n = [-\beta_n^{-1}, \beta_n^{-1}] \) for any sequence \( (\beta_n) \) satisfying (b) (here we identify \( \mathbb{T} \) with \( \mathbb{R} \)). The case \( G = \mathbb{T} \) is analogous.

It was shown in \([1]\) that every locally compact metric 0-dimensional abelian group \( G \) will admit such families; here we take \( (\beta_n) \) to be any sequence of positive numbers smaller than 1 that satisfy (b) for some \( \mu \in (0, 1) \), and put

\[ T_n = A(\Gamma, (x \in G: d(x, 0) < \beta_n)). \]

Other examples of groups admitting families \( \{T_n\}_{n=1}^{\infty}, (\beta_n) \) as above will be obtained from finite products and homomorphic images of 0-dimensional groups and the real line. In general for finite products we have

**Theorem 2.** Suppose \((G, d), (G', d')\) are metric LCA groups that admit families \( \{T_n\}_{n=1}^{\infty}, (\beta_n) \), \( \{T'_n\}_{n=1}^{\infty}, (\beta'_n) \) respectively, each satisfying (a)-(c) of Theorem 1. Then \( G \times G' \) also satisfies (a)-(c) with the families \( \{T_n \times T'_n\}_{n=1}^{\infty}, (\min\{\beta_n, \beta'_n\}) \), and \( \mu'' = \max\{\mu, \mu'\} \).

**Proof.** Property (a) follows easily from

\[
|\gamma \gamma'((a, a')) - 1| = |\gamma(a)\gamma'(a') - 1| \leq |\gamma(a) - 1| + |\gamma'(a') - 1|
\]

\[
\leq C\beta_n^{-1}d(a, 0) + C'\beta_n^{-1}d(a', 0')
\]

\[
\leq (C + C')\max\{\beta_n^{-1}, \beta_n^{-1}\}d''((a, a'), (0, 0'))
\]

for \( [\gamma, \gamma'] \in [T_n, T'_n] \). The proof of (b) is trivial. For (c) we make use of the uniqueness property of Haar measure to obtain

\[
\theta''(3[T_n, T'_n]) = \theta''(3T_n, 3T'_n) = \eta\theta(3T_n)\theta'(3T'_n)
\]

\[
\leq \eta KK'\theta(T_n)\theta'(T'_n) = KK'\theta''([T_n, T'_n]),
\]

where \( \eta \) is some positive constant.

Our corresponding result for homomorphic images is a little more restrictive; here we consider \( G/H \), where \( H \) is a compact subgroup of \( G \).

**Theorem 3.** Let \( G \) be a metric LCA group that admits families \( \{T_n\}_{n=1}^{\infty}, (\beta_n) \) satisfying (a)-(c) of Theorem 1. If \( H \) is a compact subgroup of \( G \) then \( G/H \) also satisfies (a)-(c) with the families \( \{(T_n \cap A(\Gamma, H))\}_{n=1}^{\infty}, (\beta_n) \) and the same constant \( \mu \).
Proof. To show that (a) is satisfied, consider \( \gamma^+ \in (\Gamma_n \cap A(\Gamma, H))^+ \) and \( a \in G \). For any \( b \in a + H \),

\[
|\gamma^+(a + H) - 1| = |\gamma^+(b + H) - 1| = |\gamma(b) - 1| \leq C\beta_n^{-1} d(b, 0).
\]

Hence

\[
|\gamma^+(a + H) - 1| \leq C\beta_n^{-1} \inf\{d(b, 0) : b \in a + H\} = C\beta_n^{-1} d^*(a + H, H).
\]

Obviously (b) holds with the same \( (\beta_n), \mu \). For the proof of (c) we require \( H \) to be compact, so that \( A(\Gamma, H) \) is open; here, for some positive constant \( \eta \),

\[
\theta^* (3(\Gamma_n \cap A(\Gamma, H))^+) = \theta^* (3(\Gamma_n \cap A(\Gamma, H))^+) = \eta \theta(3(\Gamma_n \cap A(\Gamma, H)))
\]

\[
\leq \eta K \theta(\Gamma_n \cap A(\Gamma, H)) = K \theta^* (\Gamma_n \cap A(\Gamma, H))^+,
\]

as required. □

Our main example of a finite dimensional compact abelian group, namely the \( \alpha \)-adic solenoid \( \Sigma_\alpha \), is not covered by Theorems 2 and 3. We define \( \Sigma_\alpha \) by

\[
\Sigma_\alpha = (\mathbb{R} \times \Delta_\alpha) / B,
\]

where \( B \) is the cyclic discrete subgroup of \( \mathbb{R} \times \Delta_\alpha \) generated by \( (1, u) \), \( u = (1, 0, 0, \ldots) \).

A metric \( d^* \) will be given for \( \Sigma_\alpha \) according to the specifications in the beginning of this paper. We assert that the families \( \{T_n\}_{n=1}^\infty \) satisfy (a)–(c), where

\[
T_n = (\lbrack \lbrack -\beta_n^{-1}, \beta_n^{-1}] \cap A(\Gamma_n, \Lambda_n) \rbrack \cap A(\mathbb{R} \times \Delta_\alpha, B))^+.
\]

Now Theorems 2, 3 apply to show that (a), (b) hold. To prove that (c) holds for the above choice of \( \{T_n\}_{n=1}^\infty \), set

\[
\kappa = \theta^* (\lbrack \lbrack 0, 1 \rbrack \cap A(\Gamma_n, \Lambda_n) \rbrack \cap A(\mathbb{R} \times \Delta_\alpha, B))^+).
\]

For each \( m \in \mathbb{Z} \),

\[
\theta^* (\lbrack \lbrack (m, m + 1) \cap A(\Gamma_n, \Lambda_n) \rbrack \cap A(\mathbb{R} \times \Delta_\alpha, B))^+ = \theta^* ([m, 0]^+ + \lbrack \lbrack 0, 1 \rbrack \cap A(\Gamma_n, \Lambda_n) \rbrack \cap A(\mathbb{R} \times \Delta_\alpha, B))^+) = \kappa,
\]

where we have used the translation-invariance of \( \theta^* \) (note that \( [m, 0] \in A(\mathbb{R} \times \Delta_\alpha, B) \)). Hence

\[
\frac{\theta^* (3T_n)}{\theta^* (T_n)} \leq \frac{\theta^* (\lbrack \lbrack [-3\beta_n^{-1}, 3\beta_n^{-1}] \cap A(\Gamma_n, \Lambda_n) \rbrack \cap A(\mathbb{R} \times \Delta_\alpha, B))^+)}{\theta^* (\lbrack \lbrack [-\beta_n^{-1}, \beta_n^{-1}] \cap A(\Gamma_n, \Lambda_n) \rbrack \cap A(\mathbb{R} \times \Delta_\alpha, B))^+)}
\]

\[
\leq (6\beta_n^{-1} + 2)/(2\beta_n^{-1} - 2) < 4
\]

for \( n \) suitably large, which is all we need to prove. □

Now Theorem 4 of [3] can be modified to give the following analogue of Jackson's theorem for \( \Sigma_\alpha \):

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Theorem 4. For each $n \in \{1, 2, \ldots\}$ put
\[ \Omega_n = \left\{ \frac{l}{a_0a_1 \cdots a_{n-1}} : l \in \mathbb{Z} \text{ and } \left| \frac{l}{a_0a_1 \cdots a_{n-1}} \right| \leq \beta_n^{-1} \right\} \]
(here we are identifying the character group of $\Sigma_a$ with a subgroup of the group $\mathbb{Q}$ of rational numbers). Then there is a constant $K$ such that
\[ E_{\Omega_n}(p; f) \leq K \omega(p; f; \sigma_B((-\beta_n, \beta_n) \times \Lambda_n)) \]
for every $f \in L^p(\Sigma_a)$ if $p \in [1, \infty)$, or for every continuous $f$ if $p = \infty$.

It follows that if $f \in \text{Lip}_p \alpha$ for some $\alpha > 0$ then
\[ E_{\Omega_n}(p; f) = O(\beta_n^\alpha). \]
Combining (4) with Theorem 1, and observing that
\[ \sigma_n = (\sigma_{\mathbb{Z}}((-\beta_n, \beta_n) \times \Lambda_n), \Lambda_n) + \Lambda(B_n, \Lambda_n). \]
then, under the further assumption that $\beta_{n+1} = \mu \beta_n$ for $n = 1, 2, \ldots$, we have for the $\mathbb{a}$-adic solenoid:

Theorem 5. Let $\alpha \in (0, 1)$ be given. Then $f \in \text{Lip}_p \alpha$ if and only if $E_{\Omega_n}(p; f) = O(\beta_n^\alpha)$ where, for $p = \infty$, $f$ is taken to be continuous.

Obtaining an analogous result for finite dimensional groups is a little more involved. First we see from \cite[Lemma 1]{5} that a finite dimensional compact metric abelian group is topologically isomorphic with $(\Delta_a^\infty \times \Sigma_a^\text{dim} G)/H$, where $\mathbb{a}$ is chosen so that $\mathbb{Q}$ is the character group of $\Sigma_a$, $\Delta_a^\infty$ is the direct product of countably many copies of $\Delta_a$, $\dim G$ is the (finite) topological dimension of $G$, and $H$ is a closed 0-dimensional subgroup of $\Delta_a^\infty \times \Sigma_a^\text{dim} G$. Note that $\Delta_a^\infty \times \Sigma_a^\text{dim} G$ is compact, and hence so is $H$.

Now write
\[ U_n = \mathbb{W}_n \times (\sigma_B((-\beta_n, \beta_n) \times \Lambda_n))^{\text{dim} G}, \]
and
\[ V_n = \pi_H(U_n). \]
Let $\mathbb{W}_n$ (respectively $\mathbb{V}_n$) be the open subgroup of $\Delta_a^\infty \times \Sigma_a^\text{dim} G$ (respectively $(\Delta_a^\infty \times \Sigma_a^\text{dim} G)/H$) generated by $U_n$ (respectively $V_n$) (note that $\mathbb{W}_n = \mathbb{W}_n \times \Sigma_a^\text{dim} G$) and set
\[ \nabla_n = \sigma_n^{-1}(V_n \cap (A(\Gamma_a, \mathbb{W}_n), [\mathbb{W}_n]^\text{dim} G) \cap A(\Gamma_a, \mathbb{W}_n \cap H)) + \]\nhere we use the notation that for an open subgroup $\mathbb{W}$ and a closed subgroup $H$ of an LCA group $G$, $\sigma_\mathbb{W}$ denotes the restriction map of $\Gamma_G$ onto $\Gamma_\mathbb{W}$, and $\nu$ denotes the adjoint of the natural topological isomorphism
\[ \nu : \pi_H(\mathbb{W}) \to \mathbb{W}/(\mathbb{W} \cap H). \]
With these definitions we have, from [3, Theorems 1–3], the existence of a constant $K$ such that

$$E_{\nabla_n}(p; f) \leq K\omega(p; f; V_n)$$

for every $f \in L^p(\Delta^\infty_n \times \Sigma_n^{\dim G})/H)$ if $p \in [1, \infty)$, or for every continuous $f$ if $p = \infty$.

To match this result with Theorem 1 we require that

$$\nabla_n = (A(\Gamma^\infty_n, \mathcal{W}_n), [\Omega_n]^\dim G) \cap A(\Gamma^\infty_n \times \Sigma_n^{\dim G}, H))^+.$$  

As part of the proof of (6) we appeal to the following general result:

**Lemma.** Let $G$ be an LCA group, with a closed subgroup $H$ and an open subgroup $\mathcal{U}$. Then, for any $T \subset \Gamma_G$,

$$\sigma_{\mathcal{U}H}(\mathcal{U}H)((\mathcal{U}H) \cap A(\Gamma_G, H))^+ \subset \nu^-(\sigma_{\mathcal{U}H}(\mathcal{U}H) \cap A(\Gamma_G, \mathcal{U}H))^+).$$

**Proof.** First notice that both sides of the above inclusion are subsets of $\Gamma_{\mathcal{U}H}(\mathcal{U}H)$; just use [4, (24.5)] and the property that $\sigma_{\mathcal{U}H}(\mathcal{U}H)$ is an open subgroup of $G/H$.

Let $\chi \in \mathcal{T} \cap A(\Gamma_G, H)$ and $x \in \mathcal{U}$. Then

$$\sigma_{\mathcal{U}H}(\mathcal{U}H)(\mathcal{T} \cap A(\Gamma_G, H))^+(x + H) = \chi^+(x + H) = \chi(x).$$

Also $\sigma_{\mathcal{U}H}(\mathcal{T} \cap A(\Gamma_G, H))$ and (recall that $x \in \mathcal{U}$)

$$\nu^-(\sigma_{\mathcal{U}H}(\mathcal{T} \cap A(\Gamma_G, H))^+(x + H) = \sigma_{\mathcal{U}H}(\mathcal{T} \cap A(\Gamma_G, H))^+(x + \mathcal{U}) = \sigma_{\mathcal{U}H}(\mathcal{T} \cap A(\Gamma_G, H))^+(x) = \chi(x),$$

so that $\sigma_{\mathcal{U}H}(\mathcal{T} \cap A(\Gamma_G, H))^+(x) = \nu^-(\sigma_{\mathcal{U}H}(\mathcal{T} \cap A(\Gamma_G, H))^+)$. □

Unfortunately the inclusion reverse to that in (7) does not seem to hold in general. However we can establish (the special case) (6) quite easily as follows; consider $\eta \in \nabla_n$ and write $\eta = \eta^+$, where $\gamma \in A(\Gamma^\infty_n \times \Sigma_n^{\dim G}, H)$. We know that $\gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_d\}$ for some $\gamma_0 \in \Gamma^\infty_n$ and $\gamma_1, \ldots, \gamma_d \in \Sigma_n$ ($d = \dim G$). Since $\sigma_{\mathcal{U}H}(\mathcal{T} \cap A(\Gamma_G, H))^+(\eta)$ can be identified with an element of

$$\sigma_{\mathcal{U}H}(\mathcal{T} \cap A(\Gamma_G, H))^+(\eta) = [(0), [\Omega_n]^\dim G),$$

it is apparent that $\gamma_0 \in A(\Gamma^\infty_n, \mathcal{W}_n)$ and $\gamma_1, \ldots, \gamma_d \in \Omega_n$, that is

$$\gamma \in A(\Gamma^\infty_n, \mathcal{W}_n), [\Omega_n]^\dim G];$$

this gives the required result.

Now that we know that (6) holds we can appeal to (5) and Theorems 1, 2, and 3 (the metric on $\Delta^\infty_n$ is chosen with respect to the basis $(\mathcal{W}_n)$) to obtain, once more under the assumption that $\beta_n+1 = \mu\beta_n$ for $n = 1, 2, \ldots$:

**Theorem 6.** Let $G$ be a finite dimensional compact metric abelian group. With the notation above we have that for $\alpha \in (0, 1)$ given, $f \in \text{Lip}_p\alpha$ if and only if $E_{\nabla_n}(p; f) = O(\beta_n^\alpha)$ where, for $p = \infty$, $f$ is taken to be continuous.
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