OPEN SUBSETS OF $R^\infty$ ARE STABLE

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Abstract. Let $U$ be an open subset of $R^\infty = \text{dir lim } R^n$, where $R$ denotes the reals. We show that $U \times R^\infty$ is homeomorphic to $U$. Combined with previous work of the author we obtain the corollary that two open subsets of $R^\infty$ are homeomorphic if and only if they have the same homotopy type.

Let $R$ denote the reals, and let $R^\infty = \text{dir lim } R^n$. Here we prove the following.

Theorem 1. If $U$ is an open subset of $R^\infty$ then $U \times R^\infty$ is homeomorphic to $U$.

In [2, Theorem II-8] it is shown that if two paracompact, connected $R^\infty$-manifolds $N, M$ have the same homotopy type then $N \times R^\infty$ and $M \times R^\infty$ are homeomorphic. Since open subsets of $R^\infty$ are paracompact [2, Proposition III-1] we obtain the following.

Corollary 2. Two open subsets of $R^\infty$ are homeomorphic iff they have the same homotopy type.

Proof of Theorem 1. We regard $R^\infty$ as $\bigcup \{ R^n | n = 1, 2, \ldots \}$, and we identify $R^n$ with $R^n \times \{0\} \subset R^{n+k}$, $k > 1$, so that $R^n \subset R^{n+k} \subset R^\infty$. Let $U$ be an open subset of $R^\infty$. We may assume that $U$ is connected. By [2, Proposition III-2] (or by elementary reasoning) $U = \bigcup \{ C_n | n = 1, 2, \ldots \}$ where $C_n$ is compact, $C_n \subset C_{n+1}$, $n > 1$, and where a subset $V$ of $U$ is open in $U$ iff $V \cap C_n$ is open in $C_n$, $n > 1$. An elementary argument shows we may assume additionally that $C_n \subset R^n$.

In what follows the word manifold will be used only for a compact, p.l. (piecewise linear) manifold, possibly with boundary. We observe that if $K$ is any compact set and if $K \subset W$ where $W$ is open in $R^n$, then there is an $n$-manifold $M$ such that $K \subset M \subset W$. This manifold may be obtained, for example, by first finding a polyhedron in $W$ containing $K$ and then taking a regular neighborhood of this polyhedron in $W$. For $n > 1$ and $\varepsilon > 0$ let

$$D(n, \varepsilon) = \{ x = (x_1, \ldots, x_n) \in R^n | |x_i| \leq \varepsilon, i = 1, \ldots, n \text{ and } x_n \geq 0 \}.$$ 

For $n > 1$ let $U^n = U \cap R^n$. 

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To begin, choose a 2-manifold $M_1$ such that $C_2 \subset M_1 \subset U^2$. Choose $\delta_1 > 0$ such that $M_1 \times [0, \delta_1] \subset U^3$. Let $N_1 = \partial (M_1 \times [0, \delta_1])$, where $\partial$ denotes manifold boundary. Choose $\epsilon_1 > 0$ such that $N_1 \times D(1, \epsilon_1) \subset U^4$. Now choose a 4-manifold $M_2$ such that $[N_1 \times D(1, \epsilon_1)] \cup C_4 \subset M_2 \subset U^4$. Choose $\delta_2 > 0$ such that $M_2 \times [0, \delta_2] \subset U^5$, and let $N_2 = \partial (M_2 \times [0, \delta_2])$. Choose $\epsilon_2 > 0$ such that $N_2 \times D(3, \epsilon_2) \subset U^8$. Inductively, suppose we have chosen $N_1, \ldots, N_k$ and $\epsilon_1, \ldots, \epsilon_k$ where $N_i$ is a closed 2-manifold and $\epsilon_i > 0$, $1 \leq i \leq k$, where

$$\left[ N_{i-1} \times D \left( 2^{(i-1)} - 1, \epsilon_i \right) \right] \cup C_{2i} \subset N_i \subset U^{2i+1},$$

$2 \leq i \leq k$, and where $N_k \times D(2^k - 1, \epsilon_k) \subset U^{2^{k+1}}$. Choose a $2^{k+1}$-manifold $M_{k+1}$ such that

$$\left[ N_k \times D \left( 2^k - 1, \epsilon_k \right) \right] \cup C_{2^{k+1}} \subset M_{k+1} \subset U^{2^{k+1}}.$$ Choose $\delta_{k+1} > 0$ such that $M_{k+1} \times [0, \delta_{k+1}] \subset U^{2^{k+1}+1}$, and let $N_{k+1} = \partial (M_{k+1} \times [0, \delta_{k+1}])$. Choose $\epsilon_{k+1} > 0$ such that

$$N_{k+1} \times D \left( 2^{(k+1)} - 1, \epsilon_{k+1} \right) \subset U^{2^{k+2}}.$$ In this way we obtain $N_k$, $\epsilon_k > 0$, $k > 1$, such that $N_k$ is a closed 2-manifold and

$$\left[ N_k \times D \left( 2^k - 1, \epsilon_k \right) \right] \cup C_{2^k} \subset N_k \subset U^{2^{k+1}}.$$ For $k \geq 1$ let $E_k = \partial D(2^k - 1, \epsilon_k)$ and let $F_k = \partial D(2^k - 1, 2^k - 1)$. Then $N_k \times E_k$ and $N_k \times F_k$ are closed $(2^{(k+1)} - 2)$-manifolds and $N_k \times E_k \subset N_k \times F_k$. For $k \geq 1$ define a linear homeomorphism $h_k: N_k \times E_k \to N_k \times F_k$ by $h_k(x, y) = (x, [(2^k - 1)/\epsilon_k]y)$ where $x \in N_k$ and $y \in E_k$. Consider the diagram

$$\begin{array}{cccccc}
N_1 \times E_1 & \xrightarrow{\alpha_1} & \cdots & N_k \times E_k & \xrightarrow{\alpha_k} & N_{k+1} \times E_{k+1} & \xrightarrow{\alpha_{k+1}} & \cdots \\
\downarrow h_1 & & & \downarrow h_k & & \downarrow h_{k+1} & & \\
N_1 \times F_1 & \xrightarrow{\beta_1} & \cdots & N_k \times F_k & \xrightarrow{\beta_k} & N_{k+1} \times F_{k+1} & \xrightarrow{\beta_{k+1}} & \cdots \\
\end{array}$$

Here $\alpha_k$ is the inclusion $\alpha_k(x, y) = ((x, y), 0)$ where $x \in N_k$, $y \in E_k$, $(x, y) \in N_{k+1}$, $0 \in E_{k+1}$, and $\beta_k$ is the inclusion $\beta_k(x, y) = ((x, 0), (y, 0))$ where $x \in N_k$, $y \in F_k$, $(x, 0) \in N_{k+1}$ and $(y, 0) \in F_{k+1}$. Since

$$N_{k+1} \supset N_k \times D \left( 2^k - 1, \epsilon_k \right)$$

and since $E_k$ contracts in $D(2^k - 1, \epsilon_k)$, the p.l. embedding $h_{k+1}\alpha_k$ is easily seen to be homotopic to the map $(x, y) \mapsto ((x, 0), 0)$. Since

$$F_{k+1} = \partial D \left( 2^{k+1} - 1, 2^{k+1} - 1 \right) \supset D \left( 2^k - 1, 2^k - 1 \right)$$

and since $F_k$ contracts in $D(2^k - 1, 2^k - 1)$, the p.l. embedding $\beta_k h_k$ is easily seen to be homotopic to the map $(x, y) \mapsto ((x, 0), (0, 0))$. Thus, $h_{k+1}\alpha_k$ and $\beta_k h_k$ are homotopic, $k > 1$. License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
We proceed to inductively replace the p.l. homeomorphisms $h_k$ by p.l. homeomorphisms $g_k$ such that $g_{k+1} \alpha_k = \beta_k g_k$. Let $g_1 = h_1$. Now $N_1 \times E_1$ is a closed 2-manifold and $N_2 \times F_2$ is a closed 6 = (2(2) + 2)-manifold. By [3, Corollary 5.9, p. 65] $h_1 \alpha_1$ and $\beta_1 g_1$ are ambient isotopic. Thus, there is a level preserving p.l. homeomorphism $H_2: N_2 \times F_2 \times I \rightarrow N_2 \times F_2 \times I$ such that $(H_2)_0 = id$ and $(H_2)_1 h_1 \alpha_1 = \beta_1 g_1$. Let $g_2 = (H_2)_1 h_2$. Then $g_2$ is a p.l. homeomorphism and $g_2 \alpha_1 = \beta_1 g_1$. Also, if $\pi: N_2 \times F_2 \times I \rightarrow N_2 \times F_2$ is the projection map, then $\pi \circ H \circ (h_2 \times id): N_2 \times F_2 \times I \rightarrow N_2 \times F_2$ is a homotopy between $h_2$ and $g_2$. Thus, $\beta_2 g_2 \sim \beta_2 h_2 \sim h_2 \alpha_2$. (By "~" we denote "is homotopic to"). Suppose, inductively, that we have defined $g_1, \ldots, g_k$ such that $g_i \alpha_{i-1} = \beta_{i-1} g_{i-1}$, $i = 2, \ldots, k$, and $\beta_k g_k \sim h_k \alpha_k$. Noting that 
$$\dim(N_{k+1} \times F_{k+1}) = 2^{(k+2)} - 2 = 2^{2(k+1)} - 2 = 2 \dim(N_k \times E_k) + 2$$ and again applying [3, Corollary 5.9, p. 65] we know that $\beta_k g_k$ and $h_{k+1} \alpha_k$ are ambient isotopic. Proceeding as in the definition of $g_2$ we obtain a p.l. homeomorphism $g_{k+1}: N_{k+1} \times E_{k+1} \rightarrow N_{k+1} \times F_{k+1}$ such that $g_{k+1} \alpha_k = \beta_k g_k$ and $\beta_{k+1} g_{k+1} \sim h_{k+1} \alpha_{k+1}$. By induction we obtain our desired sequence of p.l. homeomorphisms $\{g_n\}$.

The sequence $\{g_n\}$ induces a homeomorphism
$$g_\infty: \operatorname{dir \ lim}(N_k \times E_k; \alpha_k) \rightarrow \operatorname{dir \ lim}(N_k \times F_k; \beta_k).$$

Our theorem will be established if we show that $\operatorname{dir \ lim}(N_k \times E_k; \alpha_k) = U$ and $\operatorname{dir \ lim}(N_k \times F_k; \beta_k) = U \times R^\infty$. Clearly these equalities hold for the underlying point sets. Let $\mathcal{C} \subset U$ be such that $\mathcal{C} \cap (N_k \times E_k)$ is open in $N_k \times E_k$, $k \geq 1$. Given any $C_n$ we have $C_n \subset N_j$ for any $j$ such that $2^j \geq n$. Since $\mathcal{C} \cap (N_j \times E_j)$ is open in $N_j \times E_j$ we have $\mathcal{C} \cap C_n = [\mathcal{C} \cap (N_j \times E_j)] \cap C_n$ is open in $(N_j \times E_j) \cap C_n = C_n$. By choice of $(C_n)$, $\mathcal{C}$ is open in $U$. On the other hand, if $\mathcal{C}$ is open in $U$ then, since each $N_k \times E_k \subset U^{2^k+1}$ has the relative topology induced from $U$, $\mathcal{C} \cap (N_k \times E_k)$ is open in $N_k \times E_k$, $k \geq 1$. Thus, $U = \operatorname{dir \ lim}(N_k \times E_k; \alpha_k)$. For the second direct limit note that $U \times R^\infty$, being homeomorphic to an open subset of $R^\infty$ (since $R^\infty \times R^\infty \cong R^\infty$ by [1, Corollary III-1]), is a connected paracompact $R$-manifold. By [2, Proposition III-2] $U \times R^\infty = \cup \{K_n|n = 1, \ldots, \infty\}$ where each $K$ is compact, and where $V \subset U \times R^\infty$ is open iff $V \cap K_n$ is open in $K_n$, all $n \geq 1$. Let $V \subset U \times R^\infty$ be such that $V \cap (N_k \times F_k)$ is open in $N_k \times F_k$, all $k$. Given $K_n$ we can choose $j$ (e.g. see [1, Lemma III-6]) such that
$$K_n \subset \pi_U(K_n) \times \pi_{R^\infty}(K_n) \subset C_{2^j} \times D(2^{(j-1)} - 2, 2^{(j-1)} - 2) \subset C_{2^j} \times \partial D(2^{(j-1)} - 1, 2^{(j-1)} - 1) \subset N_j \times F_j.$$

Since $V \cap (N_j \times F_j)$ is open in $N_j \times F_j$ we then have $V \cap K_n = [V \cap (N_j \times F_j)] \cap K_n$ is open in $(N_j \times F_j) \cap K_n = K_n$. Thus, $V$ is open in $U \times R^\infty$. Conversely, if $V$ is open in $U \times R^\infty$ then, since each $N_k \times F_k$ has the relative topology induced from $U \times R^\infty$, $V \cap (N_k \times F_k)$ is open in $N_k \times F_k$, $k \geq 1$.

It follows that $U \times R^\infty = \operatorname{dir \ lim}(N_k \times F_k; \beta_k)$, and our proof is complete.
Bibliography


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