ON THE COHOMOLOGY GROUPS OF A MANIFOLD WITH A NONINTEGRABLE SUBBUNDLE

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Abstract. We define the cohomology groups \( H^*(M, \mathfrak{g}^{\mathfrak{e}}) \) of a manifold \( M \) with a nonintegrable subbundle \( E \), and we give the condition for the existence of a bundle-like metric with respect to \( E \).

1. Introduction. N. Abe [1], J. L. Heitsch [3] and I. Vaisman [6] studied some cohomology groups of a manifold with a foliation ("integrable case"). In this note, we generalize their results to the nonintegrable case, that is, we define the cohomology groups \( H^*(M, \mathfrak{g}^{\mathfrak{e}}) \) of a manifold \( M \) with a nonintegrable subbundle \( E \). In the case that \( M \) is a riemannian manifold, by its cohomology groups, we give the condition for the existence of a bundle-like metric with respect to \( E \).

2. Preliminaries. We shall be in \( C^\infty \)-category. Let \( M \) be an \( n \)-dimensional paracompact manifold with tangent bundle \( TM \). Let \( E \) be a subbundle of \( TM \) with the constant fibre dimension \( n - p \) \((0 < p < n)\). We assume that \( E \) is not integrable. \( \Gamma(\cdot) \) denotes the functor associating to a bundle its vector space of sections, and \([\ , \ ]\) the bracket operator on \( \Gamma(TM) \). Let \( C(E) \) be the "Cauchy characteristic subbundle" of \( E \), i.e. the fibre \( C_x(E) \) over \( x \in M \) of \( C(E) \) consists of \( X_x \in E_x \) (= the fibre over \( x \) of \( E \)) such that \([X, Y]_x \in E_x \) for any \( Y_x \in E_x, X, Y \in \Gamma(E) \), and, for all \( x \in M \), \( \dim C_x(E) \) is assumed to be constant. Then \( C(E) \) is an integrable subbundle of \( E \) (naturally, of \( TM \)). We assume that the fibre dimension of \( C(E) \) is \( n - q \) \((0 < p < q < n)\). We set
\[
\begin{align*}
Q &= TM/E, \\
E' &= E/C(E),
\end{align*}
\]
and, by a suitable riemannian metric on \( TM \), we have isomorphisms
\[
\begin{align*}
\Gamma(TM) &= \Gamma(Q) \oplus \Gamma(E), \\
\Gamma(TM) &= \Gamma(TM/C(E)) \oplus \Gamma(C(E)), \\
\Gamma(TM) &= \Gamma(Q) \oplus \Gamma(E') \oplus \Gamma(C(E)).
\end{align*}
\]

3. \((s, t, u)\)-forms and cohomology groups \( H^*(M, \mathfrak{g}^{\mathfrak{e}}) \). Let \( A' \) be the space of all \( r \)-forms on \( M \) and \( d \) the exterior derivative.

Definition. An \( r \)-form \( \omega \in A' \) is a \((s, t, u)\)-form, if
(i) \( s + t + u = r \), and

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(ii) $\omega(X_1, \ldots, X_r) = 0$ except for $s$ arguments $X_1$'s in $\Gamma(Q)$, $t$ arguments $X_1$'s in $\Gamma(E)$ and $u$ arguments $X_1$'s in $\Gamma(C(E))$.

Let $A^{s,t,u}$ be the space of all $(s, t, u)$-forms on $M$, and we have a decomposition

$$A^r = \sum_{s + t + u = r} A^{s,t,u}.$$

By the definition of $C(E)$, we have

$$(3) \quad [\Gamma(C(E)), \Gamma(E)] \subset \Gamma(E), \quad [\Gamma(C(E)), \Gamma(C(E))] \subset \Gamma(C(E)).$$

From this, we have that the partial derivative

$$\hat{\partial}: A^{s,t,u} \to A^{s,t,u+1}$$

induced by the exterior derivative satisfies $(\hat{\partial})^2 = 0$. Let $\mathcal{A}^{s,t,u}$ be the sheaf of germs of $(s, t, u)$-forms. Then each $\mathcal{A}^{s,t,u}$ is a fine sheaf. Let $\mathcal{A}^{s,t}$ be the sheaf defined by $\mathcal{A}^{s,t} = \mathcal{A}^{s,t,0} \cap \ker(\hat{\partial})$.

**Remark.** $\mathcal{A}^{0,0}$ denotes the sheaf of germs of functions which are constants on the leaves of $C(E)$.

**Theorem 1.** There exists a fine resolution of the sheaf

$$\varphi_{s,t,u}: 0 \to \mathcal{A}^{s,t,0} \to \mathcal{A}^{s,t,1} \to \mathcal{A}^{s,t,2} \to \cdots,$$

where $i$ denotes the natural inclusion.

**Proof.** Since we can obtain the Poincaré lemma for the operator $\hat{\partial}$ and the $(s, t, u)$-form on the open unit ball in euclidean $n$-space (cf. [6]), we easily prove the assertion of the theorem.

Let $H^u(M, \mathcal{A}^{s,t})$ be the cohomology groups of $M$ with coefficients in the sheaf $\mathcal{A}^{s,t}$. Then we have

**Theorem 2.** There exist isomorphisms

$$H^0(M, \mathcal{A}^{s,t}) \cong A^{s,t,0} \cap \ker(\hat{\partial}),$$

$$H^u(M, \mathcal{A}^{s,t}) \cong A^{s,t,u} \cap \ker(\hat{\partial})/\hat{\partial}A^{s,t,u-1} \quad \text{for } u > 1.$$

**Corollary 3.** $H^u(M, \mathcal{A}^{s,t}) = \{0\}$ for $s > p$ or $t > q - p$ or $u > n - q$.

4. Generalized Bott connection and cohomology groups $H^u(M, \mathcal{A}^{s,t}(Q))$.

Let $\pi: TM \to Q = TM/E$ be the canonical projection. We define a map

$$\hat{\nabla}: \Gamma(C(E)) \times \Gamma(Q) \to \Gamma(Q)$$

by

$$\hat{\nabla}(X(S)) = \pi_*([X, \tilde{S}])$$

for $\forall X \in \Gamma(C(E)), \forall S \in \Gamma(Q)$ and $\tilde{S} \in \Gamma(TM)$ such that $\pi_*\tilde{S} = S$.

From (3), this is well defined. Let $\nabla'$ be any connection on $Q$. For $X \in \Gamma(TM)$, from (2), we can write $X = X_1 + X_2, \ X_1 \in \Gamma(C(E)), \ X_2 \in \Gamma(TM/C(E))$. Thus we define a map

$$\nabla: \Gamma(TM) \times \Gamma(Q) \to \Gamma(Q)$$
by

$$V_X(S) = \nabla_X(S) + \nabla_X'(S).$$

Then $V$ is a connection on $Q$, and is called a generalized Bott connection (cf. [2], [4]).

Let $A^{s,t,u}(Q)$ be the space of all $Q$-valued $(s, t, u)$-forms on $M$ and $\mathfrak{A}^{s,t,u}(Q)$ the corresponding sheaf. $\hat{\delta}$ operating on $A^{s,t,u}(Q)$ is given by $\nabla$. Then, as above, we have a fine resolution of the sheaf $\mathfrak{A}^{s,t}(Q) = \mathfrak{A}^{s,t,0}(Q) \cap \ker(\hat{\delta})$. Thus we have

**Theorem 4.** There exist isomorphisms:

$$H^0(M, \mathfrak{A}^{s,t}(Q)) \cong A^{s,t,0}(Q) \cap \ker(\hat{\delta}),$$

$$H^u(M, \mathfrak{A}^{s,t}(Q)) \cong A^{s,t,u}(Q) \cap \ker(\hat{\delta})/\delta A^{s,t,u-1}(Q) \quad \text{for } u \geq 1.$$ 

**Remark.** If $E$ is integrable, the same results are given by J. L. Heitsch [3].

5. **Bundle-like metric with respect to $E$.** Let $M$ be an $n$-dimensional riemannian manifold with the metric $g$, and $E^\perp$ the orthogonal complement of $E$ in $TM$. Let $\hat{\nabla}: \Gamma(C(E)) \times \Gamma(E^\perp) \rightarrow \Gamma(E^\perp)$ be a map defined by $\hat{\nabla}_X(S) = \pi_a([X, S])$ ($\pi: TM \rightarrow E^\perp$ the canonical projection), we define a connection $\nabla$ on $E^\perp$ as in (5).

**Definition.** The riemannian metric $g$ is a bundle-like metric with respect to $E$, if $(\nabla_X g)(S_1, S_2) = 0$ for $\forall X \in \Gamma(C(E)), \forall S_1, \forall S_2 \in \Gamma(E^\perp)$.

In the following, we assume that the fibre dimension of $E$ is $n - 1$. Let $\{e_A\}$ be an orthonormal frame such that $e_1 \in \Gamma(E^\perp)$ and $e_a \in \Gamma(E)$, and $\{\omega^A\}$ its dual ($1 < A < n, 2 < a < n$). We assume that $\omega^1$ is a global form (if necessary, we assume that $E$ is transversally orientable).

**Lemma 5.** $g$ is a bundle-like metric with respect to $E$ if and only if $\nabla_X(e_1) = 0$ for $\forall X \in \Gamma(C(E))$.

**Proof.** For $\forall S_1 = \xi \cdot e_1, \forall S_2 = \eta \cdot e_1 \in \Gamma(E^\perp)$ ($\xi, \eta$: functions),

$$(\nabla_X g)(S_1, S_2) = X(\nabla_X g(S_1, S_2)) - g(\nabla_X(S_1), S_2) - g(S_1, \nabla_X(S_2))$$

$$= X(\xi \cdot e_1) - \eta \cdot X(\xi) - (\xi \cdot \eta) g(\nabla_X(e_1), e_1)$$

$$= X(\xi \cdot \eta) - (\xi \cdot \eta) g(e_1, \nabla_X(e_1))$$

$$= -2(\xi \cdot \eta) g(e_1, \nabla_X(e_1)).$$

Thus we have the assertion of the lemma.

By the above metric $g$, we have an isomorphism $\Gamma(Q) \cong \Gamma(E^\perp)$, and we can identify the connections $\nabla$ on $Q$ and on $E^\perp$.

**Lemma 6.** $\hat{\delta}_1 = 0$ if and only if $\nabla_X(e_1) = 0$ for $\forall X \in \Gamma(C(E))$.

**Proof.** For $\forall S = \xi \cdot e_1$ ($\xi$: function),
\[ \hat{\omega}^1(S, X) = -X(\omega^1(S)) - \omega^1([S, X]) = -X(\xi) + \omega^1(\nabla_X(S)) \]
\[ = -X(\xi) + X(\xi) + \xi \cdot \omega^1(\nabla_X(e_1)) \]
\[ = \xi \cdot \omega^1(\nabla_X(e_1)). \]

Thus we have the assertion of the lemma.

From the above lemmas, we have

**Theorem 7.** Let \( M \) be an \( n \)-dimensional riemannian manifold with the metric \( g \) and \( E \) a nonintegrable, transversally orientable subbundle of \( TM \) of fibre dimension \( n - 1 \). If \( H^0(M, \mathcal{D}^{1,0}) \cong A^{1,0} \), then \( g \) is a bundle-like metric with respect to \( E \). Conversely, if \( g \) is a bundle-like metric with respect to \( E \), then \( H^0(M, \mathcal{D}^{1,0}) \neq \{0\} \).

**Proof.** A nonzero 1-form \( \omega^1 \) is a \((1, 0, 0)\)-form on \( M \). If \( H^0(M, \mathcal{D}^{1,0}) \cong A^{1,0} \), then \( A^{4,0} \cap \ker(\hat{\omega}^1) \) and we have \( \hat{\omega}^1 = 0 \). By Lemmas 5 and 6, \( g \) is a bundle-like metric with respect to \( E \). Conversely, if \( g \) is a bundle-like metric with respect to \( E \), by Lemmas 5 and 6, we have \( \hat{\omega}^1 = 0 \) and \( \omega^1 \) is a nonzero \((1, 0, 0)\)-form on \( M \). Thus we have \( H^0(M, \mathcal{D}^{1,0}) \neq \{0\} \).

**Remark.** In the case that \( E \) is integrable, \( M \) is compact and \( g \) is a bundle-like metric, then \( H^1(M, R) \neq \{0\} \) (cf. R. Sacksteder [5]).

**References**


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