ANALYTIC TOEPLITZ OPERATORS WITH AUTOMORPHIC SYMBOL. II

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ABSTRACT. For \( \phi \) in \( H^\infty \), let \( T_\phi \) be the analytic Toeplitz operator with symbol \( \phi \) and let \( \{ T_\phi \}' \) be the commutant of \( T_\phi \). Two infinite Blaschke products \( \phi \) and \( \psi \) are exhibited such that \( \{ T_\phi \}' \cap \{ T_\psi \}' \) is not equal to \( \{ T_\theta \}' \) for any inner function \( \theta \). Also, two questions on reducing subspaces of analytic Toeplitz operators are answered.

1. Introduction. For \( \phi \) in \( H^\infty \) of the unit disk, the analytic Toeplitz operator \( T_\phi \) on \( H^2 \) is defined by \( T_\phi (f) = \phi f \). The commutant of \( T_\phi \) is the set of operators \( S \) on \( H^2 \) such that \( ST_\phi = T_\phi S \) and is denoted \( \{ T_\phi \}' \). In [4], Deddens and Wong ask the following question.

**Question 1.** Suppose \( \{ \chi_\alpha : \alpha \in \mathcal{A} \} \) is a family of inner functions. Is \( \bigcap_{\alpha \in \mathcal{A}} \{ T_{\chi_\alpha} \}' \) equal to \( \{ T_\theta \}' \) where \( \theta \) is some inner function of which each \( \chi_\alpha \) is a function?

James Thomson has shown that if one of the \( \chi_\alpha \) is a finite Blaschke product, then the answer to Question 1 is affirmative [13]. In this paper it is shown that Thomson's result is sharp. In fact, we produce two infinite Blaschke products \( \phi \) and \( \psi \) such that \( \{ T_\phi \}' \cap \{ T_\psi \}' \) does not equal \( \{ T_\theta \}' \) for any inner function \( \theta \).

The second author has raised the following two questions on reducing subspaces of analytic Toeplitz operators [3].

**Question 2.** If \( \{ \chi_\alpha : \alpha \in \mathcal{A} \} \) is a collection of inner functions, if \( \chi \) is the greatest common divisor of the \( \chi_\alpha \), and if \( \mathcal{M} \) is a closed subspace of \( H^2 \) which reduces each \( T_{\chi_\alpha} \), must \( \mathcal{M} \) reduce \( T_\chi \)?

**Question 3.** If \( \phi \) in \( H^\infty \) has inner-outer factorization \( \phi = \chi F \) and if \( \mathcal{M} \) is a closed subspace of \( H^2 \) which reduces \( T_\phi \), must \( \mathcal{M} \) reduce \( T_\chi \) and \( T_F \)?

It is shown that the status of Question 2 is the same as that of Question 1: the answer is affirmative if one of the \( \chi_\alpha \) is a finite Blaschke product and there is a pair of infinite Blaschke products for which the answer is negative. The answer to Question 3 is also shown to be negative.

The counterexample for Question 1 makes use of the theory of bundle shifts developed by the first author and R. G. Douglas [2] and applied...
previously to analytic Toeplitz operators by the first author [1]. It also makes use of a result of Rudin [9] and Stout [12] on inner generators of the space of rational functions on an annulus. The proof of the affirmative part of the answer to Question 2 uses the aforementioned result of Thomson [13] and the counterexamples for Questions 2 and 3 make use of a composition operator as in [1]. Here, the composition operator is preceded by a multiplication operator that makes the product unitary and the reducing subspace is the range of a projection in the $W^*$-algebra generated by this unitary. This technique of perturbing a composition operator to make it unitary was suggested to the first author several years ago by R. G. Douglas.

2. Automorphic inner functions. Let $D$ denote the unit disk \( \{ z: |z| < 1 \} \), let $R$ denote the annulus \( \{ z: \frac{1}{2} < |z| < 1 \} \), and let $\pi$ be the holomorphic universal covering map from $D$ onto $R$ as defined in [11] and [1] by

$$
\pi(z) = \exp \left( \frac{i}{\pi} \log \frac{1}{2} \log \frac{1 + z}{1 - z} + \frac{1}{2} \log \frac{1}{2} \right)
$$

where $\log$ is the principal branch of the logarithm. It is shown in [1] that $T_\pi$ is a pure subnormal operator with spectrum contained in the closure of $R$ and normal spectrum contained in the boundary of $R$ and thus, by [2, Theorem 11], there is a vector bundle $E$ over $R$ such that the bundle shift $S_E$ is unitarily equivalent to $T_\pi$. The bundle shift $S_E$ is multiplication by $z$ on the space $H_2^2(R)$ of $H^2$ cross-sections of the bundle $E$. Let $A(R)$ be the space of continuous functions on the closure of $R$ that can be approximated uniformly by rational functions with poles off the closure of $R$ and for $\phi$ in $A(R)$ let $T_\phi^E$ be the operator on $H_2^2(R)$ defined by $T_\phi^E(f) = \phi f$. It is easily verified that $\phi(T_\pi) = T_\phi \circ \pi$ and $\phi(S_E) = T_\phi^E$ for all $\phi$ in $A(R)$. This establishes the following lemma.

**Lemma 2.1.** There is a unitary operator $V$ from $H^2(D)$ onto $H_2^2(R)$ such that $VT_\phi^E = T_\phi^E V$ for all $\phi$ in $A(R)$.

A generating set for $A(R)$ is a subset $G$ of $A(R)$ such that the smallest uniformly closed subalgebra of $A(R)$ containing $G$ is all of $A(R)$. The space $H^\infty(R)$ is the Banach algebra of all bounded analytic functions on $R$. For a set of operators $S$, the second commutant of $S$ is the commutant of the commutant of $S$ and is denoted $S''$.

**Lemma 2.2.** If $G$ is a generating set for $A(R)$, then the Banach algebra \( \{ T_\phi^E \circ \pi : \phi \in G \}'' \) is isomorphic to $H^\infty(R)$.

**Proof.** By Lemma 2.1, the algebra \( \{ T_\phi^E \circ \pi : \phi \in G \}'' \) is unitarily equivalent to the algebra \( \{ T_\phi^E : \phi \in G \}'' \). Since $G$ is a generating set for $A(R)$, the latter algebra is equal to the second commutant of the bundle shift $S_E$. The result now follows from [2, Theorem 4].

The function in $A(R)$ is said to be inner if it is unimodular on the boundary of $R$. In the following lemma and elsewhere in this paper, a
function which is a Blaschke product times a scalar of unit modulus shall be referred to as a Blaschke product.

**Lemma 2.3.** If \( \phi \) is a nonconstant inner function in \( A(R) \), then \( \phi \circ \pi \) is an infinite Blaschke product.

**Proof.** The covering map \( \pi \) is continuous on the set \( \{ z : |z| < 1, z \neq 1, z \neq -1 \} \) and maps the sets \( \{ z : |z| = 1, \text{Im } z > 0 \} \) and \( \{ z : |z| = 1, \text{Im } z < 0 \} \) onto the outer and inner boundaries of \( R \) respectively. It follows that \( \phi \circ \pi \) is an inner function of the form

\[
\phi(\pi(z)) = \lambda B(z) \exp\left(-a \frac{1+z}{1-z}\right) \exp\left(-b \frac{1+z}{-1-z}\right)
\]

where \( |\lambda| = 1, B \) is a Blaschke product, and \( a \) and \( b \) are nonnegative real numbers. If \( a \) is not equal to zero, then

\[
0 = \lim_{x \to 1^+} \phi(\pi(x)) = \lim_{t \to \infty} \phi(e^{it}/\sqrt{2}).
\]

(Here one uses the fact that \( \pi \) maps the interval \((-1, 1)\) around the circle \( \{|z| = 1/\sqrt{2}\} \) an infinite number of times.) It follows that \( \phi(e^{it}/\sqrt{2}) = 0 \) for all \( t \), hence, the function \( \phi \) is identically zero, a contradiction. Thus, \( a = 0 \) and a similar argument shows that \( b = 0 \). Thus, \( \phi \circ \pi \) is a Blaschke product. Since \( \phi \) is not constant, there is a point \( \beta \) in \( R \) such that \( \phi(\beta) = 0 \) and therefore \( \phi \circ \pi \) must vanish on the infinite set \( \pi^{-1}(\beta) \). Thus, \( \phi \circ \pi \) is an infinite Blaschke product.

**Theorem 1.** There are two infinite Blaschke products \( \phi \) and \( \psi \) such that the Banach algebra \( \{ T_\phi, T_\psi \}'' \) is isomorphic to \( H^\infty(R) \).

**Proof.** It has been shown by Rudin that there are two inner functions \( \phi_1 \) and \( \psi_1 \) which form a separating pair for \( A(R) \) [9]. It follows that the map \( z \to (\phi_1(z), \psi_1(z)) \) is an embedding of the closure of \( R \) into the closure of the polydisc \( U^2 = \{(z_1, z_2) : |z_1| < 1, |z_2| < 1\} \) which takes the boundary of \( R \) into the boundary of \( U^2 \). Let \( V \) denote the range of this embedding. Stout has shown that every function \( f \) in \( A(R) \), when viewed as a function on \( V \), can be extended to a continuous function \( \tilde{f} \) on the closure of \( U^2 \) which is analytic on \( U^2 \) [12, Theorem II.1]. Let \( \{ p_n \} \) be a sequence of polynomials in two variables which converges uniformly to \( \tilde{f} \) on the closure of \( U^2 \). Then the sequence \( \{ p_n(\phi_1, \psi_1) \} \) converges uniformly to \( f \) on the closure of \( R \) which proves that \( \phi_1 \) and \( \psi_1 \) generate \( A(R) \). Set \( \phi = \phi_1 \circ \pi \) and \( \psi = \psi_1 \circ \pi \). The theorem now follows from Lemmas 2.2 and 2.3.

To see that Theorem 1 settles Question 1, suppose that \( \theta \) is an inner function such that \( \{ T_\theta \}' \cap \{ T_\phi \}' = \{ T_\theta \}' \). Then \( \{ T_\phi, T_\psi \}'' = \{ T_\theta \}'' \). However, since \( T_\theta \) is a unilateral shift, its double commutant \( \{ T_\theta \}'' \) is isomorphic to the Banach algebra \( H^\infty(D) \). Thus, according to Theorem 1, the Banach algebras \( H^\infty(R) \) and \( H^\infty(D) \) are isomorphic. This is a contradiction. In fact, if \( R_1 \) and \( R_2 \) are two bounded domains in the plane with analytic boundary and if \( H^\infty(R_1) \) is isomorphic to \( H^\infty(R_2) \), then \( R_1 \) and \( R_2 \) are conformally
3. Reducing subspaces. We begin with an affirmative result.

**Theorem 2.** If \( \{ \chi_{a} : \alpha \in \mathcal{A} \} \) is a collection of inner functions which contains a finite Blaschke product, if \( \chi \) is the greatest common divisor of the \( \chi_{a} \), and if \( \mathcal{M} \) is a closed subspace which reduces each \( T_{\chi_{a}} \), then \( \mathcal{M} \) reduces \( T_{\chi} \).

**Proof.** The hypotheses imply that there is a finite Blaschke product \( \theta \) and inner functions \( \psi_{a} \) such that \( \chi_{a} = \psi_{a} \circ \theta \) and \( \{ T_{\chi_{a}} : \alpha \in \mathcal{A} \} = \{ T_{\theta} \} \) [13]. Set \( \Psi = \text{g.c.d.}(\psi_{a}) \). It follows that \( \chi = \Psi \circ \theta \). In other words

\[
(\ast) \quad [\text{g.c.d.}(\psi_{a})] \circ \theta = \text{g.c.d.}(\psi_{a} \circ \theta : \alpha \in \mathcal{A}).
\]

For the case here where the collection \( \{ \psi_{a} \} \) contains a finite Blaschke product, the function \( \Psi \) is the Blaschke product vanishing precisely at the common zeroes (counting multiplicities) of the \( \psi_{a} \). Since \( \theta \) is also finite Blaschke, \( \Psi \circ \theta \) is the finite Blaschke product which vanishes precisely at the common zeroes (counting multiplicities) of \( \psi_{a} \circ \theta \), and hence \( \Psi \circ \theta = \chi \). The general case of \((\ast)\) can be shown using Theorem 1 (iv) of [3]. Now suppose that \( P \) is a projection which commutes with each \( T_{\chi_{a}} \). Then \( P \) commutes with \( T_{\theta} \) and hence with \( \Psi(T_{\theta}) \). But \( \Psi(T_{\theta}) = T_{\Psi \circ \theta} = T_{\chi} \) which proves the theorem.

Let \( A \) be the linear fractional transformation which generates the covering group for \( \pi \) as in [1]. Thus, \( A \) maps the disk onto itself and a function \( \phi \) on the disk is of the form \( \Psi \circ \pi \) if and only if \( \phi \) is automorphic with respect to \( A \), that is, \( \phi(A(z)) = \phi(z) \) for all \( z \) in \( D \). The following lemma deals with a composition operator defined with respect to \( A \) which is perturbed in such a way to make the result unitary. It also deals with functions modulus automorphic with respect to \( A \). We now define these objects.

The composition operator \( C_{A} \) on \( H^{2} \) is defined by the equation \( C_{A}(f) = f \circ A \). It has been shown by Nordgren [7] that for \( f \) in \( H^{2} \),

\[
\| C_{A}(f) \|^{2} = \frac{1}{2\pi} \int_{0}^{2\pi} |f(e^{i\theta})|^{2} \text{Re} \left[ \frac{e^{i\theta} + A(0)}{e^{i\theta} - A(0)} \right] d\theta.
\]

Thus, if \( k_{A} \) is the outer function such that

\[
|k_{A}(e^{i\theta})|^{2} = \left[ \text{Re} \left( \frac{e^{i\theta} + A(0)}{e^{i\theta} - A(0)} \right) \right]^{-1},
\]

then \( k_{A} \) is an invertible element of \( H^{\infty} \) and \( \| C_{A}(k_{A}f) \|^{2} = \| f \|^{2} \) for all \( f \) in \( H^{2} \). Thus, the operator \( U_{A} \) on \( H^{2} \) defined by \( U_{A}(f) = C_{A}(k_{A}f) \) is unitary. Let \( \phi \) be in \( H^{\infty} \) of the disk, and let \( \lambda \) be a scalar of modulus one. The function \( \phi \) is said to be modulus automorphic with respect to \( A \) of index \( \lambda \) if \( \phi(A(z)) = \lambda \phi(z) \) for all \( z \) in \( D \).

**Lemma 3.1.** If \( \phi \) is modulus automorphic with respect to \( A \) of index \( \lambda \), then
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U_AT_\phi = \lambda T_\phi U_A. Hence, the operator U_A commutes with T_\phi if and only if \phi is automorphic with respect to A.

PROOF. Evaluate.

Actually, Lemma 3.1 holds for an arbitrary linear fractional transformation which maps the disk onto itself.

THEOREM 3. There are two infinite Blaschke products \phi and \psi and a subspace \mathcal{M} of H^2 such that \mathcal{M} reduces T_\phi and T_\psi and \mathcal{M} does not reduce T_\chi where \chi is the greatest common divisor of \phi and \psi.

PROOF. For a in R, let \phi_a be the Blaschke product for the set \pi^{-1}(a). It has been shown by Sarason that \phi_a is modulus automorphic with respect to A of index \frac{e^{2\pi \tau}}{\log|a|/\log 2} [11, p. 18]. Thus, if a = 1/\sqrt{2}, then the Blaschke products \phi = \phi_a\phi_a and \psi = \phi_a\phi_{ia} are automorphic with respect to A and their greatest common divisor \chi = \phi_a is modulus automorphic with respect to A of index \pi. Thus, by Lemma 3.1, the unitary operator U_A commutes with T_\phi and T_\psi and does not commute with T_\chi. Since the projections in a W*-algebra always generate the algebra, there is a projection \mathcal{P} in the W*-algebra generated by U_A such that \mathcal{P} does not commute with T_\chi. But this projection does commute with T_\phi and T_\psi which proves the theorem.

The following theorem is closely related to Theorem 3 of [1].

THEOREM 4. If \phi(z) = \pi(z) - \frac{3}{4} and if \phi = \chi F is the inner-outer factorization of \phi, then there is a reducing subspace for T_\phi which reduces neither T_\chi nor T_F.

PROOF. The function \chi is modulus automorphic and not automorphic with respect to A (see the proof of Theorem 3 in [1]). By Lemma 3.1, the unitary operator U_A does not commute with T_\chi and thus there is a projection \mathcal{P} in the W*-algebra generated by U_A such that \mathcal{P} does not commute with T_\chi. Since \phi is automorphic with respect to A, the operator U_A commutes with T_\phi by Lemma 3.1, and thus \mathcal{P} commutes with T_\phi. From the equations (1) T_\phi = T_\chi T_F, (2) T_\phi \mathcal{P} = T_\mathcal{P} T_\phi, (3) T_\chi \mathcal{P} \neq T_\mathcal{P} T_\chi, and the fact that (4) T_F is invertible, it follows that T_F \mathcal{P} \neq T_\mathcal{P} T_F. This completes the proof of Theorem 4.

4. Comments. The examples in Theorems 3 and 4 involve projections in the W*-algebra generated by U_A (sometimes called spectral projections for U_A). In fact, the operator U_A is a bilateral shift of infinite multiplicity [5] and therefore the W*-algebra generated by U_A is L^\infty of the unit circle.

The spectral subspaces for U_A are reducing subspaces for T_\phi, a fact which gives a proof of Theorem 2 in [1] that does not invoke the theory of bundle shifts. The proof of the following proposition involves an analysis of the bundle E of §2 and is omitted. Following Rosenthal [8], an operator A is said to be completely reducible if for each nonzero reducing subspace \mathcal{M}, the operator A|\mathcal{M} has a nontrivial reducing subspace.

PROPOSITION 4.1. The operator T_\phi is completely reducible.

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This proposition suggests the following reformulation of a question of Nordgren [6] which was shown to be false in general by the first author [1].

**Question.** If $\phi$ is in $H^\infty$ and if $T_\phi$ has a nontrivial reducing subspace $\mathcal{M}$ such that $T_\phi|\mathcal{M}$ is irreducible, must there be a function $\psi$ in $H^\infty$ and an inner function $\theta$ which is not a linear fractional transformation such that $\phi = \psi \circ \theta$?

In an abstract of his dissertation, Carl Cowen has announced an affirmative answer to Question 1 if for some $w$ in the unit disk, the greatest common divisor of $\{\chi_\alpha - \chi_\alpha(w)\alpha \text{ in } \mathbb{C}\}$ is finite Blaschke. It follows that Theorem 2 remains true under this assumption.

The authors wish to acknowledge the referee for simplifying the proof of Theorem 5 and for pointing out the recent results of Lubin on the operator $U_A$.

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