A REDUCIBILITY CONDITION FOR RECURSIVENESS

PAUL H. MORRIS

Abstract. A result due to Jockusch, equating recursiveness of a set to a reducibility condition on its jump, is sharpened.

Introduction. Unexplained notation is taken from Rogers [5]. In an unpublished proof in 1970, Carl Jockusch showed that if $A' \leq_{btt} \emptyset'$ then $A$ is recursive (the converse is immediate). A short proof of an indirect nature was later obtained by Gordon Phillips, a student of Jockusch. This paper gives a fairly direct proof of a more basic result from which that of Jockusch follows immediately.

We write $A \oplus B$ for the set $\{2x: x \in A\} \cup \{2x + 1: x \in B\}$. Following Soare [6] set $H_A = \{e: We \cap A \neq \emptyset\}$. In the context of $A$ co-r.e. Soare has called $H_A$ the "weak jump" of $A$. For general $A$ it seems appropriate to give this name to $H_A \oplus H_A$ (if $A$ is co-r.e. and nonempty then $H_A = H_A \oplus H_A$). The relationship of the weak jump and $S$-reducibility [2] is analogous to that of the jump and Turing reducibility. $H_A$ has been studied by Hay [3], [4] and Soare [6], [7] and has been involved in a number of interesting relationships.

Let $t$ be the tt-condition $\langle \langle x_1, \ldots, x_n \rangle, \alpha \rangle$ (see [5, p. 110]). We denote the associated set $\{x_1, \ldots, x_n\}$ by $F_t$. If $\alpha(0, \ldots, 0) = 0$ we say $t$ is zero-preserving.

Results.

Theorem 1. If $A$ is r.e. and $H_A \leq_{btt} \emptyset'$ then $A$ is recursive.

Proof. Let $n$ be the least integer such that $H_A \leq_{btt} \emptyset'$ with norm bounded by $n$. Let $h$ be a recursive function such that $e \in H_A \iff$ the tt-condition $h(e)$ is satisfied by $\emptyset'$, and each $h(e)$ has norm bounded by $n$. Assume $A$ nonrecursive. Define

$$W_{f(e,x)} = \begin{cases} W_e & \text{if } x \in \emptyset', \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that if $x \in \emptyset'$ then $f(e,x) \in H_A \iff e \in H_A$. Define $W_{g(e,y)} = W_e \cup \{y\}$. If $y \in A$ then $g(e,y) \in H_A \iff e \in H_A$. Fix $e$. Set
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\[ B_1 = \{ f(e,x) : x \in \emptyset' \text{ and } F_{h(f(e,x))} \cap \emptyset' \neq \emptyset \}, \]

\[ B_2 = \{ g(e,y) : y \in A \text{ and } F_{h(g(e,y))} \cap \emptyset' \neq \emptyset \}. \]

Note that \( B_1 \) and \( B_2 \) are r.e. Put \( B = B_1 \cup B_2 \). We will show that \( B \) is nonempty.

Observe that for any \( u \), if \( F_{h(u)} \cap \emptyset' = \emptyset \), then \( u \in H_A \Leftrightarrow h(u) \) is not zero-preserving. We distinguish two cases:

\textit{Case 1.} \( e \in H_A \). Then \( x \in \emptyset' \Leftrightarrow f(e,x) \in H_A \). Thus if \( F_{h(f(e,x))} \cap \emptyset' = \emptyset \), we have \( x \in \emptyset' \Leftrightarrow h(f(e,x)) \) is zero-preserving.

Suppose \( B_1 = \emptyset \). Then \( F_{h(f(e,x))} \cap \emptyset' \neq \emptyset \Rightarrow x \in \emptyset' \). Putting these together we get \( x \in \emptyset' \Leftrightarrow F_{h(f(e,x))} \cap \emptyset' \neq \emptyset \) or \( h(f(e,x)) \) is zero-preserving. This implies \( \emptyset' \) is r.e., a falsehood. Thus \( B_1 \neq \emptyset \).

\textit{Case 2.} \( e \notin H_A \). In this case \( y \in A \Leftrightarrow g(e,y) \in H_A \). Consequently if \( F_{h(g(e,y))} \cap \emptyset' = \emptyset \), we have \( y \in A \Leftrightarrow h(g(e,y)) \) is not zero-preserving. Assuming \( B_2 = \emptyset \) now gives \( y \in A \Leftrightarrow F_{h(g(e,y))} \cap \emptyset' \neq \emptyset \) or \( h(g(e,y)) \) is zero-preserving. It follows that \( A \) is recursive, contrary to supposition. Here we conclude \( B_2 \neq \emptyset \).

Now let \( z \) be the first element in an enumeration of \( B \). Since \( F_{h(z)} \cap \emptyset' \neq \emptyset \), we can form a tt-condition \( t \) with norm bounded by \( n - 1 \) such that \( t \) is satisfied by \( \emptyset' \Leftrightarrow h(z) \) is satisfied by \( \emptyset' \Leftrightarrow e \in H_A \).

Redefining \( h(e) = t \) we see that \( H_A \leq_{btt} \emptyset' \) with norm bounded by \( n - 1 \), contradicting the minimality of \( n \). We conclude that \( A \) is recursive. Q.E.D.

Note that the above proof does not supply a decision procedure for \( A \). Theorem 1 confirms a conjecture of Hay [3].

**Theorem 1.** If \( H_A \oplus H_{\overline{A}} \leq_{btt} \emptyset' \), then \( A \) is recursive.

**Proof.** If \( H_A \oplus H_{\overline{A}} \leq_{btt} \emptyset' \), then \( A \leq_{btt} \emptyset' \).

By [5, Theorem 14-IX] \( A \) is a Boolean combination of r.e. sets. It follows from Ershov [1] that there are r.e. sets \( R_1, \ldots, R_n \) such that \( R_1 \subseteq \cdots \subseteq R_n \) and

\[ A = \begin{cases} \bigcup_{i=1}^{n/2} (R_{2i} - R_{2i-1}) & \text{if } n \text{ is even,} \\ R_1 \cup \bigcup_{i=1}^{(n-1)/2} (R_{2i+1} - R_{2i}) & \text{if } n \text{ is odd.} \end{cases} \]

We will prove the theorem by induction on \( n \). For \( n = 1 \) the result follows from Theorem 1. Suppose \( n > 1 \). Let \( f \) enumerate \( R_n \). Let \( B = f^{-1}(\overline{A}) \). Then \( B \leq_m A \). Hence \( H_B \leq_{btt} H_{\overline{A}} \) and \( H_B \leq_{btt} H_A \) and so \( H_B \oplus H_B \leq_{btt} \emptyset' \).

Let \( S_i = f^{-1}(R_i), 1 \leq i \leq n - 1 \). Then

\[ B = \begin{cases} \bigcup_{i=1}^{(n-1)/2} (S_{2i} - S_{2i-1}) & \text{if } n - 1 \text{ is even,} \\ S_1 \cup \bigcup_{i=1}^{(n-2)/2} (S_{2i+1} - S_{2i}) & \text{if } n - 1 \text{ is odd.} \end{cases} \]

The inductive hypothesis now yields \( B \) is recursive. It follows that \( A \) is r.e. By Theorem 1, \( A \) is recursive. Q.E.D.
Corollary (Jockusch). If $A' \leq_{\text{bit}} \emptyset'$ then $A$ is recursive.

Proof. Clearly $H_A \leq_1 A'$ and $H_A \leq_1 A'$. The result follows.

Closing remarks. In view of Theorem 2, it might be supposed that $H_A \leq_{\text{bit}} \emptyset' \leftrightarrow A \text{ r.e.}$ This is false, as is demonstrated by an elaborate construction in [3].

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References


Department of Computer Science, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

Current address: 2901 Hillegass #2, Berkeley, California 94705