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ANOTHER PROOF OF DOUBLY INVARIANT SUBSPACE THEOREM

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ABSTRACT. The purpose of this note is to give a new proof of the doubly invariant subspace theorem in the context of weak-* Dirichlet algebras without Hilbert space methods.

Recall that by definition a weak-* Dirichlet algebra is an algebra A of essentially bounded measurable functions on a probability measure space $(X, \mathcal{Q}, m)$ such that (i) the constant functions lie in A; (ii) $A + \bar{A}$ is weak-* dense in $L^\infty(m)$ (the bar denotes conjugation, here and always); and (iii) for all f and g in A, $\int_X fg\, dm = \int_X f\, dm \int_X g\, dm$.

Recall that by definition a doubly invariant subspace is a closed (weak-* closed if $p = \infty$) subspace of $L^p(m)$ which is invariant under multiplication by functions in $A + \bar{A}$.

THEOREM. *Let M be a doubly invariant subspace of $L^p(m)$ $(0 < p \leqslant \infty)$. Then $M = \chi_E L^p(m)$ for some measurable subset E (where χ_E denotes the characteristic function of E).*

PROOF. Since $A + \bar{A}$ is weak-* dense in $L^\infty(m)$, M is invariant under multiplication by functions in $L^\infty(m)$ (i.e. $L^\infty(m)M \subseteq M$). Let E_f be the support set of f in M, i.e. the complement of a set of maximal measure on which f is null. If $f, g \in M$, there exists $h \in M$ with $E_h = E_f \cup E_g$. For since $L^\infty(m)M \subseteq M$, $(1 - \chi_{E_f})g \in M$. Let $h = f + (1 - \chi_{E_f})g$. Let E be the support set of M, i.e. the complement of a set of maximal measure on which all $f \in M$ are null. Suppose $\alpha = \sup\{m(E_f); f \in M\}$. Choose $f_n \in M$ with $m(E_{f_n}) \to \alpha$ and $E_{f_1} \subseteq E_{f_2} \subseteq \ldots$. Alter the functions f_n so that their supports are disjoint. Define

$$q_N = \frac{\left|\sum_{n=1}^N 2^{-n} f_n\right|^2}{\left|\sum_{n=1}^N 2^{-n} f_n\right|^2 + 1/N};$$

then $q_N \to \chi_E$ a.e. as $N \to \infty$ and $|q_N| < 1$ a.e. For let $f_0 = \sum_{n=1}^\infty 2^{-n} f_n$; then for $p < \infty$, by the Lebesgue convergence theorem, $f_0 \in M$ and for $p = \infty$, by bounded convergence, $f_0 \in M$. It is clear that $m(E_{f_0}) = \alpha$ and, hence, $E_{f_0} = E$. Since $L^\infty(m)M \subseteq M$ and $\sum_{n=1}^N 2^{-n} \bar{f}_n / (|\sum_{n=1}^N 2^{-n} f_n|^2 + 1/N) \in L^\infty(m)$, the

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q_n belongs to M. Thus $\chi_E \in M$ and, hence, $M = \chi_E L^p(m)$.

The proof with Hilbert space methods is well known [1], [2].

References

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2. T. P. Srinivasan, *Doubly invariant subspaces*, Pacific J. Math. **14** (1964), 701–707. MR **29** #1528.

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