A NOTE ON GREEN'S RELATIONS ON THE SEMIGROUP $N_n$

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Abstract. Solvability criteria for nonnegative matrix equations are applied in characterizing the first three of the four Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{D}$ and $\mathcal{J}$ on the semigroup $N_n$ of all $n \times n$ nonnegative matrices. For the relation $\mathcal{J}$, it is shown that $\mathcal{D} = \mathcal{J}$ when the relation is restricted to the regular matrices in $N_n$ although on the entire semigroup $N_n$, $n \geq 3$, $\mathcal{D} \neq \mathcal{J}$.

Introduction. In recent years, much research has been concerned with the development of the algebraic structure of the $n \times n$ nonnegative matrices. Topics investigated range from characterizing multiplicative groups [5] and semigroups [4], [11], [12], to initiating a theory of primes [1], [10]. A further topic which has received interest, in this regard, concerns the characterization of the Green's relations. In [6], the Green's relations on the semigroup $\Omega_n$ of the $n \times n$ doubly stochastic matrices were completely characterized. Similar characterizations were given in the semigroup $\Sigma_n$ of $n \times n$ stochastic matrices. The study of the Green's relations on the semigroup $N_n$ of $n \times n$ nonnegative matrices was begun in [7] and in [9]. The work herein is intended as a completion of that study.

As this work is a completion of previously published research, the paper will not contain a dictionary of its language. For this the reader is referred to [7].

The theory of Green's relations on $N_n$. To place the research in its proper framework, we will sketch the results of the theory up to the position held a priori this paper.

The work of [7] and [9] characterized the Green's relations on the set of regular elements in $N_n$. These characterizations are as follows.

Theorem 1. Let $A$ and $B$ be regular of rank $r$ in $N_n$. Let $P_1$ and $P_2$ be permutation matrices such that $A_1 = AP_1$ and $B_1 = BP_2$ where $A_1$ and $B_1$ have the block form $[M \ U]$ where $M$ is $n \times r$ and contains a monomial submatrix $C$ of order $r$, i.e., $C = PD$ where $D$ is a diagonal matrix with positive main diagonal and $P$ a permutation matrix. Then $A \mathcal{R} B$ if and only if there exists a matrix $Q \in N_n$ of the form $Q = \begin{pmatrix} C & K \\ 0 & 0 \end{pmatrix}$ where $C$ is $r \times r$ and monomial, such that $A_1 = B_1 Q$.

We note that $A \mathcal{E} B$ in $N_n$ if and only if $A^T \mathcal{R} B^T$ in $N_n$. Thus, our results are stated for the relation $\mathcal{R}$, the study of $\mathcal{E}$ being dual.

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Theorem 2. If $A$, $B$ and $N_n$ are regular then $A \circ B$ if and only if they have the same rank.

As a consequence of this theorem, the following corollary is obtained.

Corollary 1. If $A$, $B$ in $N_n$ are regular, then $A \circ B$ if and only if $A \mathrel{\triangleleft} B$.

Proof. Since $\mathrel{\triangleleft} \subseteq \mathrel{\preceq}$ in any semigroup, $A \circ B$ only if $A \mathrel{\triangleleft} B$. Conversely, if $A \mathrel{\triangleleft} B$ then the equations $A = X_1 BY_1$ and $B = X_2AY_2$ are solvable for $X_1$, $Y_1$, and $X_2$, $Y_2$ in $N_n$. Thus, $A$ and $B$ have the same rank and hence from Theorem 2, $A \circ B$.

From these theorems, it is seen that the tool used to characterize Green's relations for regular elements in $N_n$ is that of rank. However, this tool is more of a vector space notion and as such is too sophisticated to characterize Green's relations on $N_n$. Here, a tool more concerned with positive cones, is necessitated. This work requires the following definitions.

Let $c(A)$ denote the cone generated by the columns of $A$. Define the cone dimension of $c(A)$, denoted $d(A)$, as the number of edges of $c(A)$. Further, as in [3], define a set $T$ of column vectors in $A$ to be independent if and only if each vector in $T$ lies on an edge of $c(A)$ and no two vectors in $T$ lie on the same edge of $c(A)$. A set of column vectors of $A$ which is not independent is called dependent. Independent and dependent sets of row vectors in $A$ are defined similarly.

For $A$, $B$ in $N_n$, if $A_i$ is a dependent column in $A$ then $(BA)_i$ is a dependent column in $BA$. Hence $d(BA) \leq d(A)$. By utilizing this notion of cone dimension we now characterize the Green's relations on $N_n$. This characterization is founded on the following lemmas.

Lemma 1. Let $A$, $B$ be in $N_n$.

(i) If $A \mathrel{\triangleleft} B$ then $d(A) = d(B)$.

(ii) If $A \mathrel{\circ} B$ then $d(A) = d(B)$ and $d(A^T) = d(B^T)$.

Proof. Note that (i) follows from the definition of $\triangleleft$. For (ii), suppose $A \mathrel{\circ} C$ and $C \mathrel{\prec} B$. Then $d(A) = d(C)$ from (i). Since $C \mathrel{\triangleleft} B$, $XC = B$ and $YA = C$ for some $X$, $Y$ in $N_n$. But then, $d(B) \leq d(C)$ and $d(C) \leq d(B)$ and consequently $d(A) = d(B)$. Finally, as $A \mathrel{\circ} B$ if and only if $A^T \mathrel{\circ} B^T$, $d(A^T) = d(B^T)$.

Lemma 2. Let $A$ be in $N_n$ with $d(A) = c$. If $A'$ is any $n \times c$ submatrix of independent columns of $A$ then $A \mathrel{\circ} \begin{bmatrix} A' & 0 \end{bmatrix}$. If further, $d(A^T) = r$ and $A''$ is any $r \times c$ submatrix in $r$ independent rows and $c$ independent columns of $A$ then

$$A \mathrel{\circ} \begin{pmatrix} A'' & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. Without loss of generality suppose the $c$ independent columns are in columns $1, \ldots, c$, i.e., $A = \begin{bmatrix} A' & A_2 \end{bmatrix}$, where $A'$ is $n \times c$. It is easily verified that $A \mathrel{\circ} \begin{bmatrix} A' & 0 \end{bmatrix}$. If further, $d(A^T) = r$, then again without loss of generality, we assume the independent columns are in columns $1, \ldots, r$. Hence

$$A = \begin{pmatrix} A'' & A_2 \\ A_3 & A_4 \end{pmatrix}.$$
where $A''$ is $r \times c$. As above,

$$A \oplus \begin{pmatrix} A'' & 0 \\ A_3 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A'' & 0 \\ A_3 & 0 \end{pmatrix} \ominus \begin{pmatrix} A'' & 0 \\ 0 & 0 \end{pmatrix}.$$ 

Hence

$$A \ominus \begin{pmatrix} A'' & 0 \\ 0 & 0 \end{pmatrix}.$$ 

Based on these results, our characterization of the Green’s relations $\ominus$, $\ominus$, and $\ominus$ on $N_n$ now follows.

**Theorem 3.** Let $A$, $B$ be in $N_n$. The following statements are equivalent:

(a) $A \ominus B$,

(b) (i) $d(A) = d(B) = d$ and

(ii) given any $n \times d$ submatrix of independent columns of $A$, say $A'$, and any $n \times d$ submatrix of independent columns of $B$, say $B'$, then there is a $d \times d$ monomial matrix $X$ so that $A'X = B'$.

**Proof.** Suppose $d(A) = d$. Let $A'$ be any submatrix of $d$ independent columns of $A$. Now $A \oplus [A' \ 0]$ by Lemma 2. Similarly, if $B'$ is any submatrix of $d$ independent columns of $B$, then $B \ominus [B' \ 0]$.

Now if $A \ominus B$, then $d(A) = d(B)$ by Lemma 1. Further, from the above remarks, $A' \ominus B'$, i.e. $A'X = B'$ and $B'Y = A'$ hold for some $X$ and $Y$ in $N_d$. Hence $A'(XY) = A'$ and so $XY = I$ from which it follows that $X$ and $Y$ are monomials. Thus, (b) is obtained.

Conversely, if (b) holds, $A' \ominus B'$. Thus $[A' \ 0] \ominus [B' \ 0]$ where $[A' \ 0]$ and $[B' \ 0]$ are in $N_n$. As $A \ominus [A' \ 0]$ and $B \ominus [B' \ 0]$, (a) follows.

**Theorem 4.** Let $A$, $B$ be in $N_n$. The following statements are equivalent:

(a) $A \ominus B$,

(b) (i) $d(A) = d(B) = c$, $d(A^T) = d(B^T) = r$ and

(ii) given any $r \times c$ submatrix $A'$ in $A$ and any $r \times c$ submatrix $B'$ in $B$ lying in $r$ independent rows and $c$ independent columns of $A$ and $B$, respectively, then there are monomial matrices $X$ in $N_r$ and $Y$ in $N_c$ such that $X A' Y = B'$.

**Proof.** The argument is similar to that in Theorem 3.

Having characterized the Green’s relations on $N_n$ for $\ominus$, $\ominus$, and $\ominus$, our efforts are now turned toward $\ominus$. Our work rests on the following corollary to Theorem 4.

**Corollary.** Let $A$, $B$ be in $N_n$ and nonsingular. Then $A \ominus B$ if and only if $X A Y = B$ has monomial solutions $X$ and $Y$ in $N_n$.

Applying this corollary, we can now show that for $n \geq 3$, $\ominus \neq \ominus$ on $N_n$.

For this consider

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 1 & 1 \end{pmatrix}.$$
Then, by direct calculation,
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
\frac{1}{4} & 0 & \frac{1}{4}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
3 & 4 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 4
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
6 & 1 & 1
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
0 & 7 & 2
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
3 & 4 & 1
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
3 & 4 & 1
\end{pmatrix}
\]
Hence \( A \not\preceq B \). But, as there are no monomials \( D_1 \) and \( D_2 \) so that \( D_1 AD_2 = B \), it follows that \( \not\preceq \not\preceq \not\preceq \) on \( N_3 \).

For \( n > 3 \), consider
\[
\begin{pmatrix}
A & 0 \\
0 & I_{n-3}
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
B & 0 \\
0 & I_{n-3}
\end{pmatrix}
\]
From the above calculations, \( \not\preceq B \), yet \( A \preceq B \). Hence \( \not\preceq \not\preceq \not\preceq \) on \( N_n, n \geq 3 \).

For \( n = 2 \), the result differs. For this case we show \( \not\preceq = \not\preceq \). In this regard, suppose \( A \not\preceq B \).

Case 1. \( A \), and hence \( B \), is singular.

Singularity here implies \( A \) and \( B \) are regular elements in \( N_2 \) and so \( A \not\preceq B \).

Case 2. \( A \), and hence \( B \), is nonsingular.

By definition \( A \not\preceq B \) implies that \( X_1 \preceq Y_1 = B \) and \( X_2 \preceq Y_2 \preceq A \) for some nonsingular \( X_1, X_2, Y_1, \) and \( Y_2 \) in \( N_2 \). Thus, each of \( X_1, X_2, Y_1, \) and \( Y_2 \) has a positive diagonal. Let \( X \prec Y \) denote the property that \( x_{ij} \geq 0 \) implies \( y_{ij} > 0 \) for all \( i,j \). Then there exist permutation matrices \( P \) and \( Q \) so that \( PAQ < B \) and permutation matrices \( R \) and \( S \) so that \( RBS < A \). Thus, \( PAQ \) and \( B \) have the same 0 pattern. We again argue cases.

Case a. \( A \), and hence \( B \), has one or two zeros.

In this case, by solving equations, diagonal matrices \( D_1 \) and \( D_2 \) in \( N_2 \) may be found so that \( D_1 PAQD_2 = B \). Hence \( A \not\preceq B \).

Case b. \( A \), and hence \( B \), is positive.

In this case, as \( X_1 \preceq Y_1 = B \) and \( X_2 \preceq Y_2 \preceq A \) it follows that \( (X_2X_1)A(Y_1Y_2) = A \). Set \( X = X_2X_1 \) and \( Y = Y_1Y_2, \) i.e. \( XAY = A \). As \( (cX)A(c^{-1}Y) = A \) for any positive number \( c \), we may assume without loss of generality that \( \det X = \det Y = \pm 1 \). Suppose \( \det X = \det Y = 1 \), i.e. \( x_{11}x_{22} - x_{12}x_{21} = 1 \) and \( y_{11}y_{22} - y_{12}y_{21} = 1 \). Suppose
\[
\max \{x_{11}, x_{22}\} = x_{11} \geq 1 \quad \text{and} \quad \max \{y_{11}, y_{22}\} = y_{11} \geq 1.
\]
If either of these two inequalities is strict, the 1, 1 entry in \( XAY \) is strictly greater than \( a_{11} \), a contradiction. But now \( x_{11} = x_{22} = y_{11} = y_{22} = 1 \). Further \( x_{12} = x_{21} = y_{12} = y_{21} = 0 \) so that \( X = Y = I \). Considering all other possible cases leads to the conclusion that \( X \) and \( Y \) are monomials and so \( X_1, X_2, Y_1, \) and \( Y_2 \) are monomials, hence \( A \not\preceq B \).

In conclusion, as \( A \not\preceq B \) if and only if the equations \( XAY = B \) and \( XBY = A \) have solutions \( X_1, Y_1, X_2, Y_2 \) in \( N_n \), respectively, and as \( \not\preceq \not\preceq \not\preceq \) on \( N_n \) for
$n \geq 3$, the authors suspect that no further satisfactory characterization of $\gamma$ exists. Thus, it is felt that the characterizations of the Green's relations on $\mathcal{N}_n$ are essentially completed by this work.

**REFERENCES**


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