A STRENGTHENING OF CENTERLESSNESS IN TEICHMÜLLER THEORY

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ABSTRACT. A group-theoretic property of braid groups is used to eliminate certain hypotheses in the construction of Teichmüller space for Riemann surfaces of finite type. The result is a clarification of the geometric content of the Bers fiber space theorem.

1. **Introduction.** In [2], the Teichmüller space of Riemann surfaces of finite type is obtained by representing a certain functor. This generalizes Grothendieck's work [6] on the case of compact Riemann surfaces. Grothendieck's work is purely geometric and makes use of analysis only to identify his Teichmüller space with that of the analysts. This "standard of purity" was abandoned in [2] where the contractibility of Teichmüller space was used in two places in an essential way. This use compromised the claim that a purely geometric basis had been provided for the Bers fiber space theorem. The purpose of the current work is to rehabilitate that claim. All the results of [2] remain valid under the sole assumption that ordinary Teichmüller space (for compact Riemann surfaces) is simply connected. Even in the absence of this hypothesis, much can be salvaged. What makes this possible is a group-theoretic property that is related to the property of being without a center.

I would like to thank my colleague, Peter Stebe, for his patience in answering all my questions about group theory. In addition, the proof of Theorem 1(i) for the case of a compact surface is due to him.

2. **TNA groups.** The following definitions suffice for our purposes although Theorem 1 is true in the more general case of braid groups with signature structure (cf. [2, Definition 1.3, p. 214]).

DEFINITION. (i) If $f: V \to T$ is continuous, we define

$$F_n(V/T) = \{(p_1, \ldots, p_n) \mid p_i \in V, f(p_i) = f(p_j), \text{ all } i,j, p_i \neq p_j \text{ for } i \neq j\}.$$

(ii) The symmetric group on n letters, $\Sigma(n)$, acts on $F_n(V/T)$ by permuting entries. $B_n(V/T) = F_n(V/T)/\Sigma(n)$.

(iii) Similarly, $B_n'(V/T) = F_n(V/T)/\Sigma(n-1)$ where the permutation group acts only on the first $n-1$ entries.

(iv) If C is a topological space, let f be the constant map $C \to \{p\}$. Then we write $F_n(C) = F_n(C/\{p\})$ and similarly for $B_n(C)$ and $B_n'(C)$.

(v) A braid group of C is a fundamental group of any of the spaces of (iv).

(vi) A group is said to be TNA if every abelian normal subgroup is trivial.

THEOREM 1. *Let C be a Riemann surface of finite type of genus at least 2. Then*
(i) *The fundamental group of C is TNA.*
(ii) *The braid groups of C are TNA.*

PROOF. (i) If C has any punctures, then $\pi_1(C)$ is free of rank at least 2. Let N be an abelian normal subgroup. Since N is abelian, $N = (1)$ or N is free cyclic. If N is cyclic, it must be of infinite index and being normal, it must be free on infinitely many generators [7, Theorem 2.10, p. 104].

If C is compact then $\pi_1(C)$ is residually free [4] and centerless [7, Corollary 4.5, p. 211]. Let N be an abelian normal subgroup and suppose $a \in N$, $a \neq 1$. Since $\pi_1(C)$ is centerless, there exists a $b \in \pi_1(C)$ such that the commutator $[a, b] \neq 1$. Choose G, a free quotient of $\pi_1(C)$ in which $[a', b'] \neq 1$, where a' and b' are the images of a and b in G. Since a' and b' do not commute, rank $G \geqslant 2$. If N' denotes the image of N in G, then N' is abelian and normal and contains $a' \neq 1$, contradicting the first case.

To prove part (ii), we need a lemma.

LEMMA. *Let $1 \to H \to G \to G' \to 1$ be an exact sequence of groups where H is TNA. Suppose either*
(a) *G' is TNA or*
(b) *G' is torsion and G is torsion-free.*
Then G is TNA.

PROOF. Let N be an abelian normal subgroup of G. Then $H \cap N$ is normal in H and abelian, whence trivial. Thus h maps N isomorphically onto its image N' in G'. (h denotes the homomorphism $G \to G'$.) Under hypothesis (a), N' is abelian and normal in G'. Under hypothesis (b), N' is a torsion-free subgroup of the torsion group G'. In either case, N' (and thus N) is trivial.

Returning to the proof of the theorem, we have exact sequences

$$1 \to \pi_1(F_n(C)) \to \pi_1(B_n(C)) \to \Sigma(n) \to 1,$$

$$1 \to \pi_1(F_n(C)) \to \pi_1(B_n'(C)) \to \Sigma(n-1) \to 1,$$

and $\pi_1(B_n(C))$ and $\pi_1(B_n'(C))$ are torsion-free [3, pp. 117–118].

Thus it suffices to prove the result for $\pi_1(F_n(C))$.

By [3, Theorem 3, p. 114], we have the exact sequence

$$1 \to \pi_1(C') \to \pi_1(F_{n+1}(C)) \to \pi_1(F_n(C)) \to 1$$

where C' denotes C with n distinct points deleted. (We shall use this notation consistently in the sequel.) Since $F_1(C) = C$, the result follows from part (i) by induction.

3. **The Bers fiber space theorem.** Let T be the Teichmüller space for compact Riemann surfaces of genus $g \geqslant 2$ and let $f: V \to T$ be the universal family of Riemann surfaces over T [6, exposé 7]. We will see that T_n, the Teichmüller space of Riemann surfaces of genus g with n punctures, is a covering space of $B_n(V/T)$ and that V'_n, the family of punctured surfaces over T_n, is a covering space of $B'_{n+1}(V/T)$. This leads to a weak form of the Bers fiber space theorem. If we adjoin the hypothesis that T is simply connected, then the theorem is true in its strong form [2, Corollary 4.3, p. 225].

We know from [6, exposé 17, §5] that the mapping $f: V \to T$ is smooth and proper and that T is a manifold so that f is topologically (even C^∞) a locally trivial fibration.

PROPOSITION. *Let C be a compact surface of genus g. Then*

$$1 \to \pi_1(B_n(C)) \to \pi_1(B_n(V/T)) \to \pi_1(T) \to 1$$

is exact and similarly for B'_n.

PROOF. Since $V \to T$ is locally trivial with fiber C, it follows that $B_n(V/T) \to T$ is locally trivial with fiber $B_n(C)$. Applying the exact homotopy sequence of a fibration, we get

$$\pi_2(B_n(V/T)) \xrightarrow{g} \pi_2(T) \to \pi_1(B_n(C)) \xrightarrow{h} \pi_1(B_n(V/T)) \to \pi_1(T) \to 1.$$

By exactness at $\pi_1(B_n(C))$, we get $\ker h \cong \operatorname{coker} g$. But as a quotient of an abelian group, $\operatorname{coker} g$ is abelian and, since it is isomorphic to $\ker h$, we have an abelian normal subgroup of $\pi_1(B_n(C))$. Since this last is TNA, $\ker h$ is trivial.

In the absence of knowledge that T is simply connected, we cannot identify $\pi_1(B_n(C))$ with $\pi_1(B_n(V/T))$. It is the identification of these two groups [2, Proposition 3.5, p. 222] which permits the conclusion that T_n is the universal covering space of $B_n(V/T)$ [2, Theorem 3.6, p. 222]. Nonetheless, [6, exposé 7, Proposition 8.3] or [2, Proposition 3.3, p. 221] implies that T_n is a covering space of $B_n(V/T)$ and the arguments of [2, §4] give analogous results for V'_n.

We summarize in the following theorem.

THEOREM 2. *T_n is a covering space of $B_n(V/T)$ and V'_n is a covering space of $B'_{n+1}(V/T)$. If T is simply connected, then T_n is the universal cover of $B_n(V/T)$ and V'_n is the covering space of $B'_{n+1}(V/T)$ associated to the subgroup $\pi_1(C')$ of $\pi_1(B'_{n+1}(V/T)) \cong \pi_1(B'_{n+1}(C))$.*

REMARK. The identification of $\pi_1(C')$ as a subgroup of $\pi_1(B'_{n+1}(C))$ is a result of the exact sequence [3, Theorem 3, p. 114]

$$1 \to \pi_1(C') \to \pi_1(B'_{n+1}(C)) \to \pi_1(B_n(C)) \to 1.$$

COROLLARY (BERS). *The universal covering space of T_{n+1} is isomorphic to the universal covering space of V'_n.*

4. **Constancy of homotopy type.** In what follows, $T_0 = T$.

THEOREM 3. $\pi_i(T_{n+1}) \cong \pi_i(T_n)$ *for all $i \geqslant 2$ and all $n \geqslant 0$. If T is simply connected then the morphism $F: T_{n+1} \to T_n$ defined by forgetting one of the punctures is a homotopy equivalence for all n.*

PROOF. Since T_{n+1} and V'_n have the same universal cover, $\pi_i(T_{n+1}) \cong \pi_i(V'_n)$ for $i \geqslant 2$. On the other hand, $f_n: V'_n \to T_n$ is a locally trivial fibration with fiber C' for all n. From the homotopy sequence of this fibration,

$$\pi_i(C') \to \pi_i(V'_n) \to \pi_i(T_n) \to \pi_{i-1}(C')$$

is exact. Since $\pi_i(C') = 0$ for $i \geqslant 2$, f_n induces an isomorphism for $i \geqslant 3$. For $i = 2$, we have

$$0 \to \pi_2(V'_n) \xrightarrow{g} \pi_2(T_n) \to \pi_1(C') \xrightarrow{h} \pi_1(V'_n),$$

where g is induced by f_n. An argument exactly like that of the Proposition shows that coker $g = 0$. Thus $\pi_i(V'_n) \cong \pi_i(T_n)$, $i \geqslant 2$. If T is simply connected, then T_{n+1} is the universal cover of V'_n so the isomorphisms $\pi_i(T_{n+1}) \cong \pi_i(V'_n)$, $i \geqslant 2$, are induced by the covering map $T_{n+1} \to V'_n$. Furthermore, $\pi_1(T_{n+1}) = \pi_1(T_n) = 1$. Thus the composite morphism $T_{n+1} \to V'_n \to T_n$ induces isomorphisms on all homotopy groups. Since the T_n are manifolds, this map is a homotopy equivalence. That the composite morphism is forgetting one puncture can be seen by [2, Proposition 4.1, p. 224].

In the geometric framework, this result corresponds to the well-known contractibility of Teichmüller spaces. The weaker statement that T is simply connected is of a different character. Grothendieck has shown [6, exposé 7, Proposition 8.5] that there exists a unique simply connected moduli space for rigidified (i.e., "marked") Riemann surfaces. Furthermore, the corresponding rigidification is uniquely determined. The contractibility of the Teichmüller space is then used only to show that this simply connected space is the Teichmüller space. The remainder of Grothendieck's program has also been achieved [6, exposé 7, Remark 3.2, 3°, p. 7-09]. Earle and Eells have shown [1, 8D, Remark 2, p. 34] that the contractibility of Teichmüller space is equivalent to that of the group of diffeomorphisms of C homotopic to the identity. A topological proof of the contractibility of this diffeomorphism group has been found by A. Gramain [5].

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