ADDENDUM TO "ARITHMETIC MEANS OF FOURIER COEFFICIENTS"

RAJENDRA SINHA

Abstract. Let $f$ be integrable and periodic with period $2\pi$. Then a necessary and sufficient condition for $\tilde{f}$ to be equivalent to a continuous function is that $-(1/w) \int \frac{f(x + u) - f(x - u)}{2 \tan(u/2)} du$ converges uniformly in $x$ as $t \to 0^+$. 

In what follows, we shall not distinguish between equivalent functions. Zamansky [2] (also see [3, Volume 1, p. 180, Exercise 5(a)]) proved the following.

Let $f$ be continuous and periodic. Then a necessary and sufficient condition for $f$ to be continuous is that

$$
\int \frac{f(x + u) - f(x - u)}{2 \tan(u/2)} du
$$

converges uniformly as $t \to 0^+$. 

In this note we show that the restriction of continuity on $f$ can be dropped in the above theorem. Equivalently we show the following.

**Theorem.** Let $f$ be integrable and periodic. Then a necessary and sufficient condition for $\tilde{f}$ to be continuous is that $\tilde{f}(x; t)$ converges uniformly in $x$ as $t \to 0^+$, where $\tilde{f}(x; t)$ is defined in (1).

**Proof.** Sufficiency is obvious since $\lim_{t \to 0^+} \tilde{f}(x; t) = \tilde{f}(x)$ for a.e. $x$.

Conversely, let $\tilde{f}$ be continuous. Take $\tilde{f} = g$ and $\tilde{f}(x; t) = g(x; t)$. Let $T_S$ and $T_H$ be defined as in [1]. Also let $g_x(t) = g(t + x)$. Now define $T_S g = T_S g_x$. Then $T_S$ is a bounded operator on $L^2$ by [1, Lemma 1.1].

Also using [1, (1.1)] with $f_x(t)$ replaced by $h_x(t) = [g_x(t) + g_x(-t)]/2$, we can show that

$$
T_x g(t) = T_S h_x(t) = T_H h_x(t) = \frac{1}{2} \left[ \pi g(x; t) + G_x(t) \right]
$$

for a.e. $t > 0$ where $G_x(t) = \sum_{n=1}^{\infty} \frac{a_n(x; g)}{n} \sin nt$.

It can easily be shown that the family $\{T_x g \mid x \in [0, 2\pi]\}$ is a normal family since $[0, 2\pi]$ is compact and

Received by the editors April 23, 1976.


*Key words and phrases.* Fourier series, conjugate function, normal family, equicontinuity.

Copyright © 1977, American Mathematical Society

243
\[ x_n \to x \Rightarrow \|g_{x_n} - g_x\|_\infty \to 0 \Rightarrow \|T_sg_{x_n} - T_sg_x\|_\infty \to 0 \]

\[ \Rightarrow \|T_{x_n}g - T_xg\|_\infty \to 0. \]

Hence, by the Ascoli-Arzela Theorem, the family \( \{T sg\mid x \in [0, 2\pi]\} \) is equicontinuous. Therefore \( \{T sg(t)\} \) converges uniformly in \( x \) as \( t \to 0^+ \).

Similarly we can show that the family \( \{G_x(t)\mid x \in [0, 2\pi]\} \) is also equicontinuous and, hence, \( G_x(t) \) converges uniformly in \( x \) as \( t \to 0^+ \). Therefore, by (2), \( g(x; t) \) converges uniformly in \( x \) as \( t \to 0^+ \). Hence the result.

**References**


**Department of Mathematics, Birla Institute of Technology and Science, Pilani, Rajasthan State, India 333031**