PRINCIPAL IDEALS IN $F$-ALGEBRAS

JAMES M. BRIGGS

Abstract. This paper is concerned with generalizations to $F$-algebras of theorems which Gleason has proved for finitely generated maximal ideals in Banach algebras. Let $A$ be a uniform commutative $F$-algebra with identity such that Spec ($A$) is locally compact; let $x$ be a nonisolated point of Spec ($A$), and let ker($x$) denote the maximal ideal of all elements of $A$ which vanish at $x$. In this paper it is shown that: If $f$ is an element of $A$ vanishing only at $x$, then the principal ideal $Af$ generated by $f$ is closed in $A$. If the polynomials in the element $f$ are dense in $A$ and if ker($x$) is finitely generated, then there exists an open set $U$ containing $x$ such that ker($y$) is generated by $f - f(y)$ for all $y$ in $U$. An example is given which shows that if $A$ is not uniform, the conclusion of the last result may not be true. In fact, the example shows that it is possible to have a nonisolated finitely generated maximal ideal in the algebra. A second example shows that in a uniform $F$-algebra with locally compact spectrum, ker($x$) can be generated by an element $f$ such that $f - f(y)$ generates no other ker($y$) even when the ker($y$) are principal.

Introduction. The results in this paper generalize to $F$-algebras results which are known for Banach algebras (see Theorem 2.1 (ii) and Theorem 2.2 of [4]). Although the results stated are only for principal maximal ideals, they should point out some of the difficulties in general for finitely generated maximal ideals.

Suppose that $B$ is a commutative Banach algebra with identity. Gleason [4] proved the following theorems dealing with the generators of an algebraically finitely generated maximal ideal.

(G1) If $I$ is a maximal ideal of $B$ which is generated by $g_1, \ldots, g_n$, then there exists a neighborhood $U$ of $I$ in the maximal ideal space of $B$ such that each maximal ideal $M$ in $U$ is generated by $g_1 - g_1(M), \ldots, g_n - g_n(M)$.

(G2) If the subalgebra generated by $1, z_1, \ldots, z_k$ is dense in $B$ and if $I$ is a finitely generated maximal ideal in $B$, then $I$ is generated by $z_1 - z_1(I), \ldots, z_k - z_k(I)$.

One immediate consequence of these theorems is that in a commutative Banach algebra with identity, the set of finitely generated maximal ideals is an open set. We will see with an example that this need not be true for $F$-algebras. Furthermore, we will show that when one considers $F$-algebras the conclusion

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of (G1) will depend on which generators one takes. The positive results given for $F$-algebras deal with the case in (G2) where $k = 1$.

**Principal ideals.** Throughout this paper $A$ will denote a uniform commutative $F$-algebra with identity such that $\text{Spec } (A)$ is locally compact. $A$ may be regarded as a (compact-open) closed subalgebra of the algebra of all continuous complex-valued functions on $\text{Spec } (A)$. A subset $K$ of $\text{Spec } (A)$ is $A$-convex if for each $x$ in $\text{Spec } (A) - K$, there is an element $f$ of $A$ such that $|f(x)| > \sup\{|f(y)| : y \in K\}$. Fix an increasing sequence $\{X_n\}$ of compact $A$-convex subsets of $\text{Spec } (A)$ such that each compact subset of $\text{Spec } (A)$ is contained in some $X_n$. Let $A$ be the completion with respect to the supremum norm on $X_n$ of the algebra $A|X_n$ of restrictions of elements of $A$ to $X_n$. Then $A = \lim\inf A_n$ and $X_n = \text{Spec } (A_n)$.

Suppose that $x$ is a nonisolated point in $\text{Spec } (A)$ and that the element $f$ in $A$ vanishes only at $x$. Define a mapping $T: g \rightarrow gf$ of $A$ into $A$. Since $x$ is not isolated, $T$ is one-to-one. The range of $T$ is $Af$, the principal ideal generated by $f$. If $T$ is not a homeomorphism of $A$ into $A$, then $f$ is called a strong topological divisor of zero (see [5, p. 47]). But then $Af$ is a closed ideal if and only if $T$ is a homeomorphism. Hence, $Af$ is a closed ideal if and only if $f$ is not a strong topological divisor of zero.

We now show that if the maximal ideal $\ker(x) = \{g \in A : g(x) = 0\}$ is finitely generated as an ideal, then the principal ideal $Af$ is closed. Notice that we do not assume that $Af$ is the maximal ideal $\ker(x)$.

If $\ker(x)$ is finitely generated, then $x$ is not in the Shilov boundary of $A$ (see Theorem 4.11 of [2]). Consequently, for large $n$, $x$ is not in $\partial A_n$, the Shilov boundary of $A_n$. Thus, by Lemma 2.2 of [3], if $\ker(x)$ is finitely generated, then $f$ is not a strong topological divisor of zero. We have proved the following theorem.

**Theorem 1.** Suppose that $x$ is a nonisolated point of $\text{Spec } (A)$ such that $\ker(x)$ is finitely generated. If $f$ is an element of $A$ vanishing only at $x$, then the principal ideal $Af$ generated by $f$ is closed.

As an immediate consequence of this theorem we get a result similar to Gleason’s theorem (G2).

**Corollary 2.** Suppose that the polynomials in the element $g$ are dense in $A$ and that $x$ is a nonisolated point in $\text{Spec } (A)$ such that $\ker(x)$ is finitely generated. Then $\ker(x)$ is principal and is generated by $g - g(x)$.

**Proof.** By replacing $g$ by $g - g(x)$ we may assume that $g(x) = 0$. Since, with this assumption, $g$ vanishes only at $x$, the principal ideal $Ag$ generated by $g$ is closed. But it is easy to see that $Ag$ is dense in $\ker(x)$. Consequently $\ker(x) = Ag$.

**Corollary 3.** If $\ker(x) = Af$, then the principal ideals $Af^n$ are closed for $n \geq 1$. 

Proof. Because $\ker(x) = Af$, $f$ vanishes only at $x$ and consequently $f^n$ also vanishes only at $x$.

The rest of this paper deals with the problem of whether a generator of a principal maximal ideal must in the sense of (G1) generate nearby maximal ideals. We first give an example of an $F$-algebra in which this is not true. See [1] for details.

Example 4. For all positive integers $k$ and $n$, let

$$K_n = [-n, n], I_n = (-1/n, 1/n), \text{ and } I_{n,k} = [-\sqrt{n} + \sqrt{nk}, 1/\sqrt{n} - \sqrt{nk}].$$

Let $D$ be the algebra of all continuous, complex-valued functions on the real line $R$ which are $n$-times continuously differentiable on $I_n$ for each $n$. For a compact set $K$ in $R$, $\| \cdot \|_K$ will denote the supremum seminorm, and for positive integers $n$ and $j$, $\| \cdot \|_{n,j}$ will denote the seminorm on $D$ given by $\|f\|_{n,j} = \sum_{i=0}^{n} (1/i!)\|f^{(i)}\|_{I_{n,j}}$. With the topology on $D$ induced by these seminorms, $D$ is a semisimple, regular, commutative $F$-algebra with identity. Furthermore, $D$ contains $C^\infty(R)$, the polynomials in the coordinate function are dense in $D$, and Spec $(D) = R$. Finally, $\ker(0)$ is principal and is generated by the coordinate function. No other maximal ideal is finitely generated.

Notice that the algebra $D$ is not a uniform algebra. We return to the uniform algebra $A$.

Lemma 5. If $\ker(x)$ is principal, then there exists an open set $U$ containing $x$ and an integer $n_0$ such that $U \cap \partial A_n = \emptyset$ for $n \geq n_0$.

Proof. Suppose that $\ker(x) = Af$. By Theorem 2.6 of [3], there exists an open set $V$ containing $x$ such that $f: V \rightarrow f(V)$ is a homeomorphism onto an open disc in $C$ and $g \circ f^{-1}$ is analytic on $f(V)$ for all $g$ in $A$. Let $U$ be an open set containing $x$ such that $U$ is compact, $U \subset V$, and $f(U)$ is a closed disc; let $n_0$ be large enough so that $U \subset X_n$ for all $n \geq n_0$. If $n \geq n_0$ and $g \in A_n$, then $(g|U) \circ f^{-1}$ is in the disc algebra on the disc $f(U)$. From this the conclusion follows.

Lemma 6. If $\ker(x) = Af$ and there exists an open set $V$ containing $x$ such that $\text{hull } (f - f(y)) = \{ y \}$ for each $y$ in $V$, then there exists an open set $U$ containing $x$ such that the principal ideals $A(f - f(y))$ are closed for each $y$ in $U$.

Proof. This follows directly from the previous lemma and Lemma 2.2 of [3].

What we might like to prove for uniform $F$-algebras is: “If $\ker(x) = Af$, then there exists a neighborhood $U$ containing $x$ such that if $y \in U$, then $\ker(y) = Af(f - f(y))$. For such a neighborhood $U$, it would necessarily be true that if $y \in U$, then $\ker(f - f(y)) = \{ y \}$. But the following example shows that this can fail to be true.

Example 7. Let $A$ be the algebra of all continuous complex-valued functions on the complex plane which are analytic on the set $|z| < 1$. Define $f$ by $f(z) = z$ if $|z| < 1$ and $f(z) = 1/z$ if $|z| > 1$. Then $\ker(0) = Af$, but if
$z_0 \neq 0$, then hull $(f - f(z_0)) = \{z_0, 1/z_0\}$. Of course, there are better behaved generators of ker($0$), namely $z$. Notice that this behavior cannot happen in a Banach algebra by Gleason's results. The following theorem follows easily from our earlier results.

**Theorem 8.** If the polynomials in the element $f$ are dense in $A$, and if ker($x$) is finitely generated, then there exists an open set $U$ containing $x$ consisting of principal maximal ideals; in fact, if $y \in U$, then ker($y$) is generated by $f - f(y)$.

Example 4 shows that the hypothesis of Theorem 8 that $A$ be uniform is necessary.

**References**


Department of Mathematics, University of Nevada, Las Vegas, Nevada 89154