Abstract. This paper is concerned with generalizations to F-algebras of theorems which Gleason has proved for finitely generated maximal ideals in Banach algebras. Let $A$ be a uniform commutative F-algebra with identity such that $\text{Spec } (A)$ is locally compact; let $x$ be a nonisolated point of $\text{Spec } (A)$, and let $\ker(x)$ denote the maximal ideal of all elements of $A$ which vanish at $x$. In this paper it is shown that: If $f$ is an element of $A$ vanishing only at $x$, then the principal ideal $Af$ generated by $f$ is closed in $A$. If the polynomials in the element $f$ are dense in $A$ and if $\ker(x)$ is finitely generated, then there exists an open set $U$ containing $x$ such that $\ker(y)$ is generated by $f - f(y)$ for all $y$ in $U$. An example is given which shows that if $A$ is not uniform, the conclusion of the last result may not be true. In fact, the example shows that it is possible to have a nonisolated finitely generated maximal ideal in the algebra. A second example shows that in a uniform F-algebra with locally compact spectrum, $\ker(x)$ can be generated by an element $f$ such that $f - f(y)$ generates no other $\ker(y)$ even when the $\ker(y)$ are principal.

Introduction. The results in this paper generalize to F-algebras results which are known for Banach algebras (see Theorem 2.1(ii) and Theorem 2.2 of [4]). Although the results stated are only for principal maximal ideals, they should point out some of the difficulties in general for finitely generated maximal ideals.

Suppose that $B$ is a commutative Banach algebra with identity. Gleason [4] proved the following theorems dealing with the generators of an algebraically finitely generated maximal ideal.

(G1) If $I$ is a maximal ideal of $B$ which is generated by $g_1, \ldots, g_n$, then there exists a neighborhood $U$ of $I$ in the maximal ideal space of $B$ such that each maximal ideal $M$ in $U$ is generated by $g_1 - \hat{g}_1(M), \ldots, g_n - \hat{g}_n(M)$.

(G2) If the subalgebra generated by $1, z_1, \ldots, z_k$ is dense in $B$ and if $I$ is a finitely generated maximal ideal in $B$, then $I$ is generated by $z_1 - \hat{z}_1(I), \ldots, z_k - \hat{z}_k(I)$.

One immediate consequence of these theorems is that in a commutative Banach algebra with identity, the set of finitely generated maximal ideals is an open set. We will see with an example that this need not be true for F-algebras. Furthermore, we will show that when one considers F-algebras the conclusion...
of (G1) will depend on which generators one takes. The positive results given for F-algebras deal with the case in (G2) where \( k = 1 \).

**Principal ideals.** Throughout this paper \( A \) will denote a uniform commutative F-algebra with identity such that Spec \((A)\) is locally compact. \( A \) may be regarded as a (compact-open) closed subalgebra of the algebra of all continuous complex-valued functions on Spec \((A)\). A subset \( K \) of Spec \((A)\) is \( A \)-convex if for each \( x \) in Spec \((A) - K \), there is an element \( f \) of \( A \) such that \( |f(x)| > \sup\{|f(y)|: y \in K\} \). Fix an increasing sequence \( \{X_n\} \) of compact \( A \)-convex subsets of Spec \((A)\) such that each compact subset of Spec \((A)\) is contained in some \( X_n \). Let \( A \) be the completion with respect to the supremum norm on \( X_n \) of the algebra \( A|X_n \) of restrictions of elements of \( A \) to \( X_n \). Then \( A = \lim\inf A_n \) and \( X_n = \text{Spec } (A_n) \).

Suppose that \( x \) is a nonisolated point in Spec \((A)\) and that the element \( f \) in \( A \) vanishes only at \( x \). Define a mapping \( T: g \to gf \) of \( A \) into \( A \). Since \( x \) is not isolated, \( T \) is one-to-one. The range of \( T \) is \( Af \), the principal ideal generated by \( f \). If \( T \) is not a homeomorphism of \( A \) into \( A \), then \( f \) is called a strong topological divisor of zero (see [5, p. 47]). But then \( Af \) is a closed ideal if and only if \( T \) is a homeomorphism. Hence, \( Af \) is a closed ideal if and only if \( f \) is not a strong topological divisor of zero.

We now show that if the maximal ideal \( \ker(x) = \{g \in A: g(x) = 0\} \) is finitely generated as an ideal, then the principal ideal \( Af \) is closed. Notice that we do not assume that \( Af \) is the maximal ideal \( \ker(x) \).

If \( \ker(x) \) is finitely generated, then \( x \) is not in the Shilov boundary of \( A \) (see Theorem 4.11 of [2]). Consequently, for large \( n \), \( x \) is not in \( \partial A_n \), the Shilov boundary of \( A_n \). Thus, by Lemma 2.2 of [3], if \( \ker(x) \) is finitely generated, then \( f \) is not a strong topological divisor of zero. We have proved the following theorem.

**Theorem 1.** Suppose that \( x \) is a nonisolated point of Spec \((A)\) such that \( \ker(x) \) is finitely generated. If \( f \) is an element of \( A \) vanishing only at \( x \), then the principal ideal \( Af \) generated by \( f \) is closed.

As an immediate consequence of this theorem we get a result similar to Gleason's theorem (G2).

**Corollary 2.** Suppose that the polynomials in the element \( g \) are dense in \( A \) and that \( x \) is a nonisolated point in Spec \((A)\) such that \( \ker(x) \) is finitely generated. Then \( \ker(x) \) is principal and is generated by \( g - g(x) \).

**Proof.** By replacing \( g \) by \( g - g(x) \) we may assume that \( g(x) = 0 \). Since, with this assumption, \( g \) vanishes only at \( x \), the principal ideal \( Ag \) generated by \( g \) is closed. But it is easy to see that \( Ag \) is dense in \( \ker(x) \). Consequently \( \ker(x) = Ag \).

**Corollary 3.** If \( \ker(x) = Af \), then the principal ideals \( Af^n \) are closed for \( n \geq 1 \).
Proof. Because \( \ker(x) = Af \) vanishes only at \( x \) and consequently \( f^n \) also vanishes only at \( x \).

The rest of this paper deals with the problem of whether a generator of a principal maximal ideal must in the sense of (G1) generate nearby maximal ideals. We first give an example of an \( F \)-algebra in which this is not true. See [1] for details.

**Example 4.** For all positive integers \( k \) and \( n \), let

\[
K_n = [-n, n], \quad I_n = (-1/n, 1/n), \quad \text{and} \quad I_{n,k} = [-\sqrt{n} + \sqrt{kn}, \sqrt{1/n} - \sqrt{kn}].
\]

Let \( D \) be the algebra of all continuous, complex-valued functions on the real line \( R \) which are \( n \)-times continuously differentiable on \( I_n \) for each \( n \). For a compact set \( K \) in \( R \), \( \| \cdot \|_K \) will denote the supremum seminorm, and for positive integers \( n \) and \( j \), \( \| \cdot \|_{n,j} \) will denote the seminorm on \( D \) given by

\[
\|f\|_{n,j} = \sum_{i=0}^n (1/i!) \|f^{(i)}\|_{I_{n,j}}.
\]

With the topology on \( D \) induced by these seminorms, \( D \) is a semisimple, regular, commutative \( F \)-algebra with identity. Furthermore, \( D \) contains \( C^\infty(R) \), the polynomials in the coordinate function are dense in \( D \), and \( \text{Spec} (D) = R \). Finally, \( \ker(0) \) is principal and is generated by the coordinate function. No other maximal ideal is finitely generated.

Notice that the algebra \( D \) is not a uniform algebra. We return to the uniform algebra \( A \).

**Lemma 5.** If \( \ker(x) \) is principal, then there exists an open set \( U \) containing \( x \) and an integer \( n_0 \) such that \( U \cap \partial A_n = \emptyset \) for \( n \geq n_0 \).

**Proof.** Suppose that \( \ker(x) = Af \). By Theorem 2.6 of [3], there exists an open set \( V \) containing \( x \) such that \( f: V \to f(V) \) is a homeomorphism onto an open disc in \( C \) and \( g \circ f^{-1} \) is analytic on \( f(V) \) for all \( g \in A \). Let \( U \) be an open set containing \( x \) such that \( \bar{U} \subset V \), and \( f(U) \) is a closed disc; let \( n_0 \) be large enough so that \( U \subset X_n \) for all \( n \geq n_0 \). If \( n \geq n_0 \) and \( g \in A_n \), then \( (g|U) \circ f^{-1} \) is in the disc algebra on the disc \( f(U) \). From this the conclusion follows.

**Lemma 6.** If \( \ker(x) = Af \) and there exists an open set \( V \) containing \( x \) such that \( \text{hull} (f - f(y)) = \{y\} \) for each \( y \) in \( V \), then there exists an open set \( U \) containing \( x \) such that the principal ideals \( A(f - f(y)) \) are closed for each \( y \) in \( U \).

**Proof.** This follows directly from the previous lemma and Lemma 2.2 of [3].

What we might like to prove for uniform \( F \)-algebras is: “If \( \ker(x) = Af \), then there exists a neighborhood \( U \) containing \( x \) such that if \( y \in U \), then \( \ker(y) = A(f - f(y)) \).” For such a neighborhood \( U \), it would necessarily be true that if \( y \in U \), then \( \text{hull} (f - f(y)) = \{y\} \). But the following example shows that this can fail to be true.

**Example 7.** Let \( A \) be the algebra of all continuous complex-valued functions on the complex plane which are analytic on the set \( |z| < 1 \). Define \( f \) by

\[
f(z) = z \quad \text{if} \quad |z| < 1 \quad \text{and} \quad f(z) = 1/z \quad \text{if} \quad |z| > 1.
\]

Then \( \ker(0) = Af \), but if
\(z_0 \neq 0\), then \(\text{hull}(f - f(z_0)) = \{z_0, 1/z_0\}\). Of course, there are better behaved generators of \(\ker(0)\), namely \(z\). Notice that this behavior cannot happen in a Banach algebra by Gleason’s results. The following theorem follows easily from our earlier results.

**Theorem 8.** If the polynomials in the element \(f\) are dense in \(A\), and if \(\ker(x)\) is finitely generated, then there exists an open set \(U\) containing \(x\) consisting of principal maximal ideals; in fact, if \(y \in U\), then \(\ker(y)\) is generated by \(f - f(y)\).

Example 4 shows that the hypothesis of Theorem 8 that \(A\) be uniform is necessary.

**References**


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