UNIQUENESS IN THE SCHAUER FIXED POINT THEOREM

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Abstract. A condition is given which guarantees the uniqueness of the fixed point in the Brouwer and Schauder fixed point theorems. The result is applied to a nonlinear boundary value problem in physiology.

1. Let $X$ be a real Banach space with a bounded convex open subset $D$, and let $F : \overline{D} \to \overline{D}$ be a continuous function which is also assumed to be compact if $X$ is infinite dimensional. The Brouwer fixed point theorem (Schauder theorem if $X$ is infinite dimensional) gives a point $x \in \overline{D}$ such that $x = F(x)$. Under the assumption that $F$ is differentiable, we give a simple condition which guarantees that the fixed point $x$ is unique. The proof is an application of degree theory. We phrase the argument for the infinite dimensional case; the reader who is interested only in the finite dimensional case may omit the compactness hypothesis.

In the last section, the result is applied to a nonlinear boundary value problem arising in physiology.

2. Suppose that $F : \overline{D} \to \overline{D}$ is compact and continuously Fréchet differentiable in $D$. Then [4, Lemma 4.1] $F'(x)$ is a compact linear operator on $X$ for each $x \in D$. Our uniqueness result is

Theorem. Let $F : \overline{D} \to \overline{D}$ be a compact continuous map which is continuously Fréchet differentiable on $D$. Suppose that (a) for each $x \in D$, 1 is not an eigenvalue of $F'(x)$, and (b) for each $x \in \partial D$, $x \neq F(x)$. Then $F$ has a unique fixed point.

In order to prove the Theorem we require a lemma. For the Lemma, recall that for any compact linear operator $A$ the spectrum, $\sigma(A)$, of $A$ consists of a countable number of points having 0 as the only possible limit point. Each nonzero $\lambda \in \sigma(A)$ is an eigenvalue of $A$. For such a $\lambda$, the null space of $(A - \lambda I)^k$ is, for all $k$ sufficiently large, independent of $k$. The dimension of this null space is called the algebraic multiplicity of the eigenvalue $\lambda$.

Lemma. Let $A$ be a compact linear operator on a real Banach space $X$. Suppose $1 \not\in \sigma(A)$, and let $\beta(A)$ denote the sum of the algebraic multiplicities of all $\lambda \in \sigma(A)$ with $\lambda$ real and $\lambda > 1$. Then there is an $\varepsilon > 0$ such that if $B$ is a compact linear operator on $X$ and $\|A - B\| < \varepsilon$, then $(-1)^{\beta(A)} = (-1)^{\beta(B)}$.

Proof of Lemma. Let $B$ be a compact operator with $\|A - B\| < \varepsilon$, where $\varepsilon$ will be determined in the course of the proof. Letting $\varepsilon < 1$, we see that
Let $Q$ be an open rectangle in the $\lambda$ plane with two sides on $\text{Re} \lambda = 1$ and $\text{Re} \lambda = M$, and the other two sides on $\text{Im} \lambda = \pm a$. Let $\Gamma$ denote the boundary of $Q$. We pick $a > 0$ so small that $\sigma(A) \cap \overline{Q}$ consists only of real eigenvalues. Then $\Gamma \cap \sigma(A) = \emptyset$. For $\epsilon$ sufficiently small, we have $\Gamma \cap \sigma(B) = \emptyset$ [2, p. 213]. Let $R_A(\lambda) = (\lambda I - A)^{-1}$ be the resolvent operator of $A$, and set

$$P_A = \int_{\Gamma} (\lambda I - A)^{-1} \, d\lambda.$$ 

Then $P_A$ is a projection whose range is the union of the eigenspaces of $A$ corresponding to the eigenvalues $\lambda \in Q \cap \sigma(A)$ [2, p. 178]. Thus, setting $d(P_A)$ = the dimension of the range of $P_A$, we find that $d(P_A)$ is the sum of the algebraic multiplicities of the eigenvalues $\lambda \in \sigma(A) \cap Q$. Defining $P_B$ in a similar way, we have a similar result for $d(P_B)$. For $\epsilon$ sufficiently small, we have $\|P_A - P_B\| < 1$, so $d(P_A) = d(P_B)$ [2, p. 33]. Since $\sigma(A) \cap Q$ consists only of real eigenvalues, and since $\lambda \in \sigma(A)$ satisfies $|\lambda| < M$, we have $d(P_A) = \beta(A)$. If $\lambda \in \sigma(B)$ is complex, then $\overline{\lambda} \in \sigma(B)$, and $\lambda$ and $\overline{\lambda}$ have the same algebraic multiplicity. Hence $d(P_B) = \beta(B) + \text{even number}$. Combining these facts, we find that $\beta(A)$ and $\beta(B)$ have the same parity, which proves the lemma.

**Proof of Theorem.** We first show that $F$ has a finite number of fixed points. For supposing otherwise, let $x^k = F(x^k)$, $k = 1, 2, \ldots$, be a sequence of fixed points. Since $F$ is compact we may, after picking a subsequence, suppose that $x^k \to x \in D$, $F(x) = x$. Hence $x \in D$. By the spectral condition, $I - F'(x)$ has a bounded inverse, so from the inverse function theorem [5, Theorem 1.20], $I - F(x)$ is $(1-1)$ in a neighborhood of $x$, contradicting $(I - F)(x^k) = 0$. Let $x^1, \ldots, x^N$ denote the fixed points of $F$. Let $U_j$ be a neighborhood of $x^j$ such that the closed sets $\overline{U}_j$ are pairwise disjoint and $\overline{U}_j \subset D$. Let $K = D \setminus \bigcup_j \overline{U}_j$, so $K$ is a closed subset of $D$ which does not contain a fixed point of $F$. Then the quantities

$$\deg(0, I - F, D) \quad \text{and} \quad \deg(0, I - F, U_j)$$

are well defined, and from the excision and additive properties of the degree [5, Proposition 3.37], we have

$$\deg(0, I - F, D) = \deg(0, I - F, D \setminus K) = \sum_j \deg(0, I - F, U_j).$$

Without loss of generality we may suppose $0 \in D$. Then $H(x, t) = x - tF(x)$, $0 < t < 1$, defines a homotopy of $F$ with the identity function $I$. Since $0 \in D$, $H(x, t) \neq 0$ for $x \in \partial D$, for otherwise $x = tF(x) + (1 - t)0 \in D$, which is a contradiction. Hence $\deg(0, I - tF, D)$ is defined, so using this homotopy,

$$\deg(0, I - F, D) = \deg(0, I, D) = 1.$$ 

From [4, Theorem 4.7],

$$\deg(I - F, 0, U_j) = (-1)^{\beta(x)}$$

where, in the notation of the Lemma, $\beta(x) = \beta(F'(x))$. Since $F'(x)$ is
continuous, we see from the Lemma that the parity of $\beta(x)$ is constant for $x \in D$. Hence $1 = \pm N$, so $N = 1$ and the fixed point is unique.

**Remarks.** (1) The same argument gives a uniqueness condition for the fixed point theorems of Altman and Rothe [5, Chapter 3]. (2) We thank Dr. John Osborn for help in proving the Lemma.

3. To illustrate our result, we consider the nonlinear boundary value problem

$$
\begin{align*}
-DC'' + (vC)' &= f(x), & 0 < x < 1, \\
C'(0) &= 0, & C(1) = a > 0, \\
\end{align*}
$$

(1)

$$
\begin{align*}
v' &= -J(x, C), & v(0) = 0.
\end{align*}
$$

(2)

This system of equations was used by Diamond and Bossert [1] to model salt and water transport in a closed-ended tube, such as a sweat gland. The diffusion coefficient $D > 0$, the function $J(x, C)$ represents an osmotic transport of water out of the tube, and the function $f(x)$ represents a source of salt into the tube. We assume that these functions are sufficiently differentiable, and that

$$
\begin{align*}
f(x) > 0, & J_C(x, C) < 0,
\end{align*}
$$

(3)

where the subscript $C$ denotes the partial derivative.

We write (1), (2) as a fixed point problem as follows. Let $C_1(x)$ be a continuous function, and with $C = C_1$, let $v(x)$ be the solution of (2). Then with $v(x)$ given, there is a unique solution $C = C_2$ of (1). To see this, we integrate (1) to obtain

$$
-DC_2'(x) + v(x)C_2(x) = f_1(x) = \int_0^x f(t) \, dt.
$$

Letting $v_1$ denote an indefinite integral of $v$, we may then solve this equation to get

$$
C_2(x) = a \exp\left\{-D^{-1}\left[v_1(1) - v_1(x)\right]\right\}
+ D^{-1} \int_x^1 f_1(t) \exp\left\{-D^{-1}\left[v_1(t) - v_1(x)\right]\right\} \, dt.
$$

(4)

We have thus defined a map $F(C_1) = C_2$ on the Banach space $X$ of continuous functions on $[0, 1]$.

The problem of solving the system (1), (2) is equivalent to the problem of finding a fixed point of $F$. From (4) and (3) we see that $C_2(x) > 0$. If $C_1(x) > 0$, we have from (3), $v(x) > w(x)$, where $w(x) = -\int_0^x f(t) \, dt$. Hence for $t > x$, $v_1(t) - v_1(x) > b(t - x)$, where $b$ is independent of $C_1(x)$. Using this in (4), we find that there is a constant $K$ such that, for any $C_1(x) > 0$, $C_2(x) < K$. If we let $D \subseteq X$ denote the convex set defined by the inequalities $0 < C(x) < K$, we have proved that $F(D) \subseteq D$. It is easy to see, using (4), that $F$ is continuous, compact, and in fact continuously Fréchet differentiable. Thus from the Schauder fixed point theorem, there is a fixed point $C = F(C)$, and hence a solution $v(x)$, $C(x)$, of (1), (2). To show that there is a unique fixed point, we must calculate the derivative of $F$. Setting $C_2 = F(C_1)$, $\hat{C}_2 = F'(C_1)\hat{C}_1$, it may be verified that $\hat{C}_1$, $\hat{C}_2$ satisfy the linear
problem
\[-D\ddot{C} + (\dot{v}C_2) + (v\dot{C}_2) = 0, \quad \dot{C}(0) = \ddot{C}(1) = 0,\]
\[\ddot{v} + J_c(x, C_1)\dot{C}_1 = 0, \quad \ddot{v}(0) = 0.\]

To apply our Theorem, we suppose $C_1 \in D$. We must verify that $1 \not\in \sigma(F'(C_1))$; that is, that $1$ is not an eigenvalue of $F'(C_1)$. Supposing the contrary, let $\check{C}$ be an eigenfunction of $F'(C_1)$ with eigenvalue $1$. Then $\check{C}_2 = \check{C}$, and hence there is a nontrivial solution $\check{v}(x)$, $\check{C}(x)$, of the problem
\[\begin{align*}
-\ddot{C} + (\dot{v}C_2) + (v\dot{C}_2) &= g_1(x), \\
\ddot{v} + J_c(x, C_1)\dot{C} &= g_2(x), \\
\ddot{C}(0) &= \ddot{C}(1) = \ddot{v}(0) = 0,
\end{align*}\]
with $g_1(x) \equiv 0$, $g_2(x) \equiv 0$. It may be verified that (5)-(7) defines a closed, densely defined operator on $(\check{v}, \check{C}) \in L_2(0, 1) \times L_2(0, 1)$, that the resolvent operator is compact, and that the adjoint operator is given by the solution of the problem
\[\begin{align*}
-\ddot{\psi} - C_2\psi &= h_1(x), \\
-\ddot{\psi} - v\dot{\psi} + J_c\dot{\psi} &= h_2(x), \\
\psi(0) &= \psi(1) = \psi(1) = 0.
\end{align*}\]

Using the Fredholm alternative and the compactness of the resolvent operator associated with the problem (5)-(7), we conclude that there are functions $\psi(x)$, $\psi(x)$, not identically zero, which satisfy (8)-(11) with $h_1(x) \equiv 0$, $h_2(x) \equiv 0$. We must have $\psi(1) \not\equiv 0$, since otherwise, by the uniqueness of the solution of the terminal value problem associated with (8), (9), we would have $\psi \equiv 0$, $\psi \equiv 0$. Let $\bar{x} < 1$ be the largest zero of $\psi'$, and suppose that $\psi'(x) > 0$ in $(\bar{x}, 1)$. Since $C_2 = F(C_1) \in D$, we have $C_2(x) > 0$ in $[0, 1]$. Hence from (8), $\bar{x}$ is the largest zero of $\psi'$, and $\psi'(x) < 0$ in $(\bar{x}, 1)$. Hence $\psi'(\bar{x}) > 0$, $\phi(\bar{x}) > 0$. Using (3), we see that the left side of (9) is $< 0$ at $x = \bar{x}$. This is a contradiction and proves that $1$ is not an eigenvalue of $F'(C_1)$. Hence by our Theorem, there is a unique fixed point of $F$, and a unique solution $v(x)$, $C(x)$ of (1), (2) with $C(x) > 0$.

**Remark.** In [3] there is given a more detailed study of osmotic flow in a tube.

**References**


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