REMARKS ON SOME FIXED POINT THEOREMS

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Abstract. A compact Hausdorff pseudo-topology is introduced on every closed convex bounded subset of a uniformly convex Banach space and is used to prove a previous theorem of the author.

In [7], we used a transfinite induction method which depends on the structure of the real line to prove the following fixed point theorem for multivalued nonexpansive mappings:

Theorem 1. Let $K$ be a closed convex bounded nonempty subset of a uniformly convex Banach space and let $T: K \to \mathcal{C}(K)$ be a nonexpansive mapping, where $\mathcal{C}(K)$ denotes the family of nonempty compact (not necessarily convex) subsets of $K$, equipped with the Hausdorff metric. Then $T$ has a fixed point, i.e. there exists $x \in K$ such that $x \in Tx$.

The properties of real numbers we used are the order property and the separability, or more explicitly, that a decreasing nonnegative transfinite sequence indexed by ordinals less than the uncountable ordinal $\Omega$ must be eventually constant.

Recently, Caristi and Kirk [1], [2], [5] have proven the following theorem and have given several interesting applications:

Theorem 2 [1], [2]. Let $X$ be a complete metric space, and let $g: X \to X$ be a self-map of $X$. Suppose that there exists a lower semicontinuous nonnegative real-valued mapping $\xi$ such that for all $x$ in $X$,

$$d(x, g(x)) \leq \xi(x) - \xi(g(x)). \quad (1)$$

Then $g$ has a fixed point.

Chi Song Wong [8] has given a simple proof of the Caristi-Kirk theorem by using the transfinite induction method mentioned above. On the other hand, we define a pseudo-compact-Hausdorff-topology on any closed convex bounded subset of a uniformly convex Banach space and give Theorem 1 a simpler and more conceptual proof. It is our feeling that the existence of such a compact Hausdorff pseudo-topology may well serve to explain the similarity between uniform convexity and compactness in some aspects of geometric fixed point theory.

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Let \((X, d)\) be a metric space. A sequence \(\{x_n\}\) in \(X\) is said to \(\Delta\)-converge to a point \(x \in X\), written \(x_n \rightarrow^\Delta x\), if
\[
\limsup_i d(x_n, x) \leq \limsup_i d(x_n, y)
\]
for every subsequence \(\{x_n\}\) of \(\{x_n\}\) and every \(y \in X\). In the terminology of asymptotic center (Edelstein [3], Lim [6]), this says that \(x\) is an asymptotic center of every subsequence of \(\{x_n\}\). \(\{x_n\}\) is said to \(\Delta\)-converge strongly to \(x\) if
\[
\lim d(x_n, x) < \liminf d(x_n, y) \quad \text{for every } y \in X.
\]
This is equivalent to saying that all subsequences of \(\{x_n\}\) have a common asymptotic center (= \(x\)) and asymptotic radius (= \(\lim d(x_n, x)\)). In general, \(x\) is not unique. That strong \(\Delta\)-convergence implies \(\Delta\)-convergence is obvious.

\(\Delta\)-convergence has the following properties of which only the first two are satisfied by strong \(\Delta\)-convergence:

(i) if \(x_n = x\) for every \(n\), then \(x_n \rightarrow^\Delta x\);
(ii) if \(x_n \rightarrow^\Delta x\), then every subsequence of \(\{x_n\}\) \(\Delta\)-converges to \(x\);
(iii) if \(\{x_n\}\) does not \(\Delta\)-converge to \(x\), then there exists a subsequence of which every subsequence does not \(\Delta\)-converge to \(x\).

Clearly these definitions and properties can be formulated for nets. A set \(X\) equipped with a convergence class satisfying (i), (ii) and (iii) (or only (i) and (ii)) will be called a pseudo-topological space and the convergence class will be called a pseudo-topology on \(X\). Note that we need only one additional axiom to define topological spaces by convergence classes, see e.g., Kelley [4]. By using nets, concepts in topological spaces can be carried over to quasi-topological spaces. Thus a quasi-topological space is compact if every net in it has a convergent subnet and is Hausdorff if a net can converge to at most one point. In what follows, (strong) \(\Delta\)-topology will refer to the quasi-topology with the convergence class given by sequences satisfying (2) (3) respectively). One may replace “sequences” by “nets” in this definition and will obtain the same conclusions of Theorems 3 and 4 below.

A metric space is said to be \(\Delta\)-complete if for every bounded sequence (or net) \(\{x_n\}\) in \(X\) there is an \(x \in X\) such that
\[
\limsup_n d(x_n, x) \leq \limsup_n d(x_n, y)
\]
for every \(y \in X\) i.e. \(\{x_n\}\) has an asymptotic center in \(X\).

**Theorem 3.** Every bounded \(\Delta\)-complete metric space \(X\) is strongly \(\Delta\)-compact (and hence \(\Delta\)-compact), i.e. every sequence in \(X\) has a strongly \(\Delta\)-convergent subsequence.

**Theorem 4.** Every closed convex bounded subset of a uniformly convex Banach space is compact Hausdorff under the (strong) \(\Delta\)-topology.

Our proofs of Theorems 3 and 4 need the following set-theoretical result.
For two sequences \(\{x_n\}\) and \(\{y_n\}\) in a set. let us say \(\{x_n\}\) is an essential
subsequence of \( \{y_n\} \) if there exists a positive integer \( N \) such that \( \{x_n\}_{n\geq N} \) is a subsequence of \( \{y_n\} \).

**Proposition 1.** Let \( X \) be a set and let \( \{x_n\} \) be a sequence in \( X \). Let \( r \) be a real-valued function whose domain is the set of subsequences of \( \{x_n\} \). Suppose \( r(y) < r(z) \) whenever \( y \) is an essential subsequence of \( z \). Then there is a subsequence \( w \) of \( \{x_n\} \) such that \( r(z) = r(w) \) for every subsequence \( z \) of \( w \).

**Proof.** Denote by \( \mathcal{F} \) the family of subsequences of \( \{x_n\} \). Define an ordering \( \leq \) on \( \mathcal{F} \) as follows:

For \( x, y \in \mathcal{F} \), we put \( x < y \) if \( x \) is an essential subsequence of \( y \) and \( r(x) < r(y) \). Then we say \( x \leq y \) if \( x < y \) or \( x \) is identically equal to \( y \).

It is easy to check that \( \leq \) is a reflexive, antisymmetric and transitive relation. Let \( C \) be a chain in \( \mathcal{F} \). Let \( r = \inf\{r(x) : x \in C\} \). If there is an \( x \in C \) such that \( r(x) = r \), then \( x \) is a lower bound for \( C \). Therefore we assume that such \( x \) does not exist. Let \( x_n \) be a sequence in \( C \) such that \( r(x_n) \) strictly decreases to \( r \). Since \( C \) is a chain, we must have \( x_1 > x_2 > \ldots \). By using the diagonal process, dropping a finite number of terms in each sequence \( X_n \) if necessary, we obtain a sequence \( y \) which is an essential subsequence of \( x_n \) for every \( n \). Then, by assumption, \( r(y) < r(x_n) \) for every \( n \) and, hence, \( r(y) < r(x) \) for every \( x \in \mathcal{F} \). Since each \( x \in C \) is an essential subsequence of \( x_n \) for some \( n \), we conclude that \( y \) is a lower bound for \( C \).

By Zorn’s lemma, \( \mathcal{F} \) has a minimal element \( z \). Let \( w \) be a subsequence of \( z \). Then \( r(w) < r(z) \). If \( r(w) < r(z) \), then \( w \leq z \) and by the minimality and antisymmetry we must have \( w = z \) which implies \( r(w) = r(z) \), a contradiction. Hence \( r(w) = r(z) \).  Q.E.D.

**Proof of Theorem 3.** Let \( \{x_n\} \) be a sequence in \( X \). For every subsequence \( \{x_{n_i}\} \), let

\[
    r(\{x_{n_i}\}) = \inf\left\{ \lim\sup_i d(x_{n_i}, y) : y \in X \right\}.
\]

By Proposition 1 \( \{x_n\} \) contains a subsequence which we still denote by \( \{x_n\} \) such that

\[
    r(\{x_{n_i}\}) = r(\{x_{n_i}\}) = r
\]

for every subsequence \( \{x_{n_i}\} \) of \( \{x_n\} \). By \( \Delta \)-completeness, there exists an \( x \in X \) such that

\[
    \lim\sup_i d(x_{n_i}, x) = r(\{x_{n_i}\}).
\]

For every subsequence \( \{x_{n_i}\} \) of \( \{x_n\} \), we have

\[
    \lim\sup_i d(x_{n_i}, x) \leq \lim\sup_i d(x_{n_i}, x) = r(\{x_{n_i}\}) = r(\{x_{n_i}\}) \leq \lim\sup_i d(x_{n_i}, x);
\]

thus

\[
    \lim\sup_i d(x_{n_i}, x) = r(\{x_{n_i}\}).
\]
This shows that all subsequences \( \{x_n\} \) of \( \{x_n\} \) have a same asymptotic center \( x \) and a same asymptotic radius \( r \). Q.E.D.

**Proof of Theorem 4.** This follows from Theorem 3 and the uniqueness of asymptotic center as proved by Edelstein [3]. Q.E.D.

Let us now give

**Proof of Theorem 1.** By a standard argument, there exist sequences \( \{x_n\} \) and \( \{y_n\} \) such that \( y_n \in Tx \) and \( \|x_n - y_n\| \to 0 \). By Theorem 4, \( \{x_n\} \) has a \( \Delta \)-convergent subsequence which we still denote by \( \{x_n\} \). Let \( x \) be its \( \Delta \)-limit. We assert that \( x \in Tx \). For each \( n \), choose \( p_n \in Tx \) such \( \|p_n - y_n\| \leq \|x - x_n\| \). Since \( Tx \) is compact, there exists a convergent subsequence \( \{p_{n_k}\} \) of \( \{p_n\} \) such that \( p_{n_k} \to p \) for some \( p \in Tx \). It can be easily shown, by using \( \|x_n - y_n\| \to 0 \), \( \|p_{n_k} - y_{n_k}\| \leq \|x - x_n\| \) and \( x_n \to \Delta x \), that \( x_{n_k} \to \Delta p \). Since also \( x_{n_k} \to \Delta x \), we must have \( x = p \in Tx \) by the uniqueness of \( \Delta \)-limit. Q.E.D.

**Remark.** I am indebted to the referee for informing me that K. Goebel has discovered independently a similar proof of Theorem 1 in a paper published in Ann. Univ. Mariae Curie-Skłodowska.

**References**


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