MONOTONE AND OPEN MAPPINGS ONTO ANR'S

JOHN J. WALSH

ABSTRACT. Let $M$ be either a compact, connected p.l. manifold of dimension at least three or a compact, connected Hubert cube manifold and let $Y$ be a compact, connected ANR (= absolute neighborhood retract). The main results of this paper are: (i) a mapping $f$ from $M$ to $Y$ is homotopic to a monotone mapping from $M$ onto $Y$ if and only if $f_*: \pi_1(M) \to \pi_1(Y)$ is surjective; (ii) a mapping $f$ from $M$ to $Y$ is homotopic to an open mapping from $M$ onto $Y$ if and only if $f_*(\pi_1(M))$ has finite index in $\pi_1(Y)$.

In [5] and [6], the author showed that a mapping $f$ from a p.l. manifold $M^m$ ($m \geq 3$) to a polyhedron $P$ is homotopic to a monotone mapping from $M$ onto $P$ if $f_*: \pi_1(M) \to \pi_1(P)$ is surjective and is homotopic to an open mapping from $M$ onto $P$ if $f_*(\pi_1(M))$ has finite index in $\pi_1(P)$. Using these results, a recent result on open mappings, and results from infinite dimensional topology, we show that the above results are true if $P$ is only assumed to be an ANR.

Terminology. We will use $M^m$ to denote either an $m$-dimensional compact, connected, p.l. (={piecewise linear}) manifold with or without boundary or, if $m = Q$, a compact, connected $Q$-manifold (where $Q$ is the Hilbert cube). By a mapping we mean a continuous function; a mapping is open if the image of each open set is open; a mapping is monotone (= UV) if each point-inverse is connected. We refer the reader to [1] for the definition of an absolute neighborhood retract (= ANR). Spaces are assumed to be separable and metric.

Main results.

Theorem 1. Let $M$ be a compact, connected, p.l. manifold with dimension at least three or a compact, connected $Q$-manifold and let $Y$ be a compact, connected ANR. A mapping $f$ from $M$ to $Y$ is homotopic to a monotone mapping from $M$ onto $Y$ if and only if $f_*: \pi_1(M) \to \pi_1(Y)$ is surjective.

Theorem 2. Let $M$ and $Y$ be as above. A mapping $f$ from $M$ to $Y$ is homotopic to an open mapping from $M$ onto $Y$ if and only if $f_*(\pi_1(M))$ has finite index in $\pi_1(Y)$.

Remark. The “only if” half of each of these theorems is well known; see...
Smale [3] and [4]. It follows from Fact 2 below that if \( f_* : \pi_1(M) \to \pi_1(Y) \) is surjective, then \( f \) is homotopic to a mapping from \( M \) onto \( Y \) which is both monotone and open.

We present below several results which will be used in the proofs of the above theorems.

**FACT 1.** Theorems 1 and 2 are true if \( Y \) is assumed to be a polyhedron or a \( Q \)-manifold. These cases are "essentially" contained in [5, Theorem 2.0] and [6, Theorem 4.0]. The theorems in [5] and [6] are stated for finite dimensional p.l. manifolds and polyhedra; however, using Chapman's result in [2] that every compact \( Q \)-manifold is homeomorphic to the product of a polyhedron and \( Q \), the proofs in [5] and [6] "work" without difficulty.

**FACT 2.** The main result in [7] is that a monotone mapping \( f \) from a compact manifold \( M^m, m \geq 3 \), onto any space \( Y \) can be homotoped (by an arbitrarily small homotopy) to a monotone open mapping \( g \) from \( M \) onto \( Y \) (with \( g^{-1}(y) \) and \( f^{-1}(y) \) having the same shape for each \( y \in Y \)). The proof in [7] "works" equally well if \( M \) is a compact \( Q \)-manifold.

**FACT 3.** A "key" step in the proofs depends on West's recent result in [8] that every compact ANR is the CE (= cell-like) image of a \( Q \)-manifold.

**PROOF OF THEOREM 1.** Let \( g: W \to Y \) be a CE mapping of a \( Q \)-manifold \( W \) onto \( Y \) and let \( \tilde{f}: M \to W \) be a "lift" of \( f \) with \( g \circ \tilde{f} \) homotopic to \( f \) (for example, let \( \tilde{f} = \iota \circ f \) where \( \iota \) is a "homotopy inverse" of \( g \)). Since \( g \) is a homotopy equivalence, we have that \( \tilde{f}_*: \pi_1(M) \to \pi_1(W) \) is surjective. Fact 1 implies that \( \tilde{f} \) is homotopic to a monotone mapping \( \tilde{h} \) from \( M \) onto \( W \); letting \( h = g \circ \tilde{h} \), \( h \) is a monotone mapping of \( M \) onto \( Y \) homotopic to \( f \).

**PROOF OF THEOREM 2.** Let \( x_0 \in M, y_0 \in Y \) with \( f(x_0) = y_0 \) and let \( p: (\tilde{Y},\tilde{y}_0) \to (Y,y_0) \) be the covering projection with

\[
p_*(\pi_1(\tilde{Y},\tilde{y}_0)) = f_*(\pi_1(M,x_0)).
\]

Since \( f_*(\pi_1(M,x_0)) \) has finite index in \( \pi_1(Y,y_0) \), we have that \( p^{-1}(y_0) \) is finite and, hence, \( \tilde{Y} \) is compact; also, \( \tilde{Y} \) is an ANR (see [1, Chapter 4, §10]). Let \( \tilde{f}: (M,x_0) \to (\tilde{Y},\tilde{y}_0) \) be a lifting of \( f \); it follows that \( \tilde{f}_*: \pi_1(M,x_0) \to \pi_1(\tilde{Y},\tilde{y}_0) \) is onto. Applying Theorem 1 and Fact 2 to \( \tilde{f} \), we have that \( \tilde{f} \) is homotopic to a monotone open mapping \( \tilde{h} \) from \( M \) onto \( \tilde{Y} \); letting \( g = p \circ \tilde{h} \), \( g \) is an open mapping from \( M \) onto \( Y \) homotopic to \( f \).

**REFERENCES**


8. J. West, *Compact ANR’s have finite type* (preprint).


Department of Mathematics, University of Oklahoma, Norman, Oklahoma 73069