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FINITE GENERATION OF CLASS GROUPS OF RINGS OF INVARIANTS

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ABSTRACT. Let R be a normal affine domain over the algebraically closed field k, and let G be a connected algebraic group acting rationally on R. It is shown that the divisor class group of R^G is a homomorphic image of an extension of a subgroup of the class group of R by a subquotient of the character group of G. In particular, if R has finitely generated class group, so does R^G.

The object of this note is to establish the following theorem: *Let R be a normal affine domain over the algebraically closed field k, and let G be a connected algebraic group acting rationally on R. Then if R has a finitely generated divisor class group, then so does R^G.* (If K is the quotient field of R, then R^G is $R \cap K^G$, so R^G is a Krull domain and hence has a divisor class group.)

The following conventions are adopted: k is the fixed algebraically closed base field. For a commutative k-algebra A, $U(A)$ denotes the group of units of A and $U_k(A) = U(A)/k^*$.

We begin with some observations regarding group actions and units.

PROPOSITION 1. *Let R be an integral domain k-algebra with quotient field K suvh that $U_k(R)$ is a finitely generated group, and let G be a connected algebraic group acting as k-algebra automorphisms of R, such that every unipotent subgroup of G acts rationally on R. Then*:

(a) *Every f in $U(R)$ is a semi-invariant for G.*

(b) *If f is in K such that $g(f)/f \in U(R)$ for all $g \in G$, then f is a semi-invariant for G.*

PROOF. First we consider the case where G is unipotent and R is the coordinate ring of the affine k-variety V. If f is a nonvanishing function on V and v an element of V, then $g \to f(gv)$ is a nonvanishing function on G, hence constant since G is unipotent. Thus f is an invariant. In general R is a direct limit of such coordinate rings, and hence every unit of R is invariant under every unipotent subgroup of G.

Now we can establish (a). We need to know that G acts trivially on $U_k(R)$, and by the above paragraph it is enough to treat the case $G = G_m$. Now $U_k(R)$ is a finitely generated free abelian group, and the action of

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G_m on $U_k(R)$ induces a homomorphism $\phi: G_m \to \mathrm{GL}_n(\mathbf{Z})$ for some n. Since G_m is divisible, so is $\phi(G_m)$. For any prime p, let K_p be the kernel of $\mathrm{GL}_n(\mathbf{Z}) \to \mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})$. Then $K_p \cap \phi(G_m)$ is of finite index in $\phi(G_m)$; since $\phi(G_m)$ is divisible this means $\phi(G_m) \subseteq K_p$. This holds for all p, so $\phi(G_m)$ is trivial, and (a) now follows.

Finally, we establish (b). For each g in G, let $g(f)/f = u(g)$. Then $u(gh) = u(g)g(u(h))$ for g, h in G. By (a), $g(u(h)) = \alpha u(h)$, for some $\alpha \in k^*$. Thus we have a homomorphism $\psi: G \to U_k(R)$ defined by letting $\psi(g)$ be the class of $u(g)$. Now $U_k(R)$ is a finitely generated free abelian group. If $G = G_a$ and k has characteristic $p > 0$, then G_a is p-torsion, hence $\psi(G_a)$ is trivial, and if k has characteristic $p = 0$, then G_a is divisible, so $\psi(G_a)$ is finitely generated free and divisible, hence trivial. If $G = G_m$, then since G_m is divisible we also have $\psi(G_m)$ trivial. Since G is generated by G_a's and G_m's, $\psi(g)$ is trivial, so $u(g) \in k$ for all g in G, and (b) follows.

The proposition has the following familiar consequence:

Corollary 2. *Let G be a connected algebraic group over k. A nonvanishing regular function on G is a constant multiple of a character of G.*

Proof. Let R be the affine coordinate ring of G. It is well known (see for example [1, p. 39]) that $U_k(R)$ is finitely generated since R is normal. If f in R is nonvanishing, part (a) of the proposition shows that for all g in G there is $X(g)$ in k^* with $f \cdot g = X(g)f$. It is clear that X is a character of G and that $f = f(e)X$.

The corollary is due to Rosenlicht [2].

The next three results are technical lemmas used in the proof of the theorem.

Lemma 3. *Let S be a Krull domain and G a group of automorphisms of S. Then every height one prime of S^G is the contraction of a height one prime of S.*

Proof. Let $R = S^G$ and let P be a height one prime of R. Choose a uniformizing parameter π for P in R. If π were a unit in $S_P = (R - P)^{-1}S$, there would be s in S and d in $R - P$ with $s\pi = d$. But since π and d are invariants, s would be also, and thus d is in P, contrary to assumption. Since π is not a unit in the Krull domain S_p, π belongs to some height one prime Q_0 of S_p, and $Q_0 \cap R_p = PR_p$. Then $Q = Q_0 \cap S$ is height one in S and $Q \cap R = P$.

Lemma 4. *Let R be a Krull domain over k and let G be a connected algebraic group over k acting rationally on R. Then every height one prime of R which contains a nonzero invariant is (set-wise) G-stable.*

Proof. Suppose the height one prime Q of R contains the nonzero invariant f. Then G permutes the finite set of height one primes containing f, and since G is connected this permutation is trivial, so Q is G-stable.

Lemma 5. *Let S be a Krull domain and G a group of automorphisms of S. Let*

f belong to the quotient field of S, and suppose that, for every height one prime Q of S, if $v_Q(f) < 0$ then $Q \cap S^G$ is nonzero. Then $f = a/b$, where $a \in S$ and $b \in S^G$.

PROOF. Let $I = \{s \in S | sf \in S\}$. Then I is a divisorial ideal of S such that $V_Q(I) = -V_Q(f)$ for all height one primes Q of S with $V_Q(f) \leqslant 0$. Write $I = Q_1^{(e_1)} \cap \cdots \cap Q_k^{(e_k)}$ where Q_i is a height one prime and $Q_i^{(e_i)}$ is the e_ith symbolic power of Q_i, each $e_i > 0$. Then $Q_1^{e_1} \cdots Q_k^{e_k}$ is contained in I, and hence $(Q_1 \cap S^G)^{e_1} \cdots (Q_k \cap S^G)^{e_k}$ is contained in I. By hypothesis, $Q_i \cap S_G$ is nonzero for each i, and hence I contains a nonzero invariant b, and this establishes the lemma.

THEOREM 6. *Let R be an affine normal domain over k, and let G be a connected algebraic group over k acting rationally on R. Then there is a group E, a surjection $E \to \mathrm{Cl}(R^G)$ and an exact sequence $1 \to F \to E \to \mathrm{Cl}(R)$, where F is a quotient of a subgroup of the character group of G.*

PROOF. We begin by defining a subgroup E_0 of $\mathrm{Div}(R)$ which will map onto E: For each height one prime P of R^G, let E_p in $\mathrm{Div}(R)$ be $E_p = \sum_{Q|P} e_Q Q$, where the sum is over the height one primes Q of R lying over P, and e_Q is the ramification index of Q. Let B denote the set of height one primes Q of R such that $Q \cap R^G$ has height at least two, and let E_0 be the subgroup of $\mathrm{Div}(R)$ generated by the E_p and B. Clearly, E_0 is a free abelian group with the E_p and B as a basis.

Define $\Phi: E_0 \to \mathrm{Div}(R^G)$ by $\Phi(E_p) = P$ and $\Phi(Q) = 0$ for $Q \in B$. By Lemma 3, Φ is a surjection. If $f \in R^G$, it is clear that $\mathrm{div}_R(f) \in E_0$ and $\Phi(\mathrm{div}_R(f)) = \mathrm{div}_{R^G}(f)$.

Now let K be the quotient field of R^G and L the quotient field of R, and let $E = E_0/\mathrm{Div}_R(K^*)$. It follows that Φ induces a surjection $E \to \mathrm{Cl}(R^G)$.

The composite $E_0 \subseteq \mathrm{Div}(R) \to \mathrm{Cl}(R)$ contains $\mathrm{Div}_R(K^*)$ in its kernel, and hence there is an induced homomorphism $E \to \mathrm{Cl}(R)$ with kernel $\mathrm{Div}_R(K^*)$. To complete the proof of the theorem we need to show that F is isomorphic to a quotient of a subgroup of the character group of G.

Let $L_0 = \{f \in L^* | \mathrm{div}_R(f) \in E_0\}$. If f is in L_0 and g is in G, then, by Lemma 4, $\mathrm{div}(g(f)) = \mathrm{div}(f)$, so $g(f)/f$ is in $U(R)$ for all g in G. By Proposition 1(b), f is a semi-invariant for G, i.e. $g \to g(f)/f$ is a character X of G. The correspondence which sends f to X is a homomorphism from L_0 to the character group of G, and f is in the kernel of this homomorphism if and only if f is an invariant. But by Lemma 5, $f = a/b$, where b is an invariant. Thus f is an invariant if and only if $f \in K$, and we have a monomorphism from L_0/K^* to the character group of G. Since L_0/K^* clearly maps onto F, the theorem is established.

COROLLARY 7. *Let R be an affine normal domain over k and let G be a connected algebraic group over k acting rationally on R.*

(a) *If $\mathrm{Cl}(R)$ is finitely generated, so is $\mathrm{Cl}(R^G)$.*

(b) *If R is factorial,* Cl (R^G) *is a homomorphic image of a subgroup of the character group of G.*

(c) *If G has no nontrivial characters,* Cl (R^G) *is a homomorphic image of a subgroup of* Cl (R).

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