AN IMPROVEMENT ON THE UPPER BOUND OF THE NILPOTENCY CLASS OF SEMIDIRECT PRODUCTS OF \( p \)-GROUPS

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Abstract. The semidirect product of a group \( A \) by a group \( B \) is necessarily nilpotent only in the case \( A \) and \( B \) are \( p \)-groups for the same prime \( p \), \( A \) is nilpotent of bounded exponent, and \( B \) is finite. In an earlier paper Morley has established an upper bound on the class of a nilpotent semidirect product of an abelian \( p \)-group of bounded exponent by an arbitrary finite \( p \)-group. In this paper this result is improved by considering a direct product decomposition for \( B \) and also by extending the results to give a new upper bound on the class in the most general case. The standard wreath product of \( A \) by \( B \) is a nilpotent semidirect product of relatively large class in the case \( A \) and \( B \) satisfy the conditions above, and this new bound improves the known results on the class of these wreath products.

1. Introduction. A group which is a semidirect product of \( A \) by \( B \) can be assumed nilpotent only with the conditions that \( A \) and \( B \) are \( p \)-groups for the same prime number \( p \), \( A \) is nilpotent of bounded exponent, and \( B \) is finite (Baumslag [1]). The standard wreath product of \( A \) by \( B \) contains all semidirect products of \( A \) by \( B \) and the exact class of \( A \wr B \) has been given by Liebeck [3] in the case \( A \) and \( B \) are abelian \( p \)-groups, \( A \) is of bounded exponent and \( B \) is finite. Meldrum [4] has given the class of \( A \wr B \) in the case \( A \) is nilpotent of exponent \( p \) and \( B \) is finite abelian. Morley [5] derives an upper bound for the class of a group which is a semidirect product of an abelian \( p \)-group of exponent \( p^{n+1} \) by an arbitrary finite \( p \)-group. In this paper, an improvement in the upper bound of [5] is established and the new bound is also extended to the most general case of a nilpotent semidirect product of \( p \)-groups. The improvement in the bound is accomplished by considering a direct product factorization of the group \( B \).

2. Notation and preliminary results. The notation and definitions used in this paper agree with, but in cases generalize, those in [5]. \((g_1, \ldots, g_n)\) and \((g_1, (n-1)g_2)\) indicate commutator elements of length \( n \), the second one having the last \( n-1 \) entries all \( g_2 \). An arbitrary ascending central series (see [2]) of the group \( G \) is denoted \( G_0 < G_1 < \cdots \) and the well-known lower
central series is written \( G(i) \), \( i = 0, 1, 2, \ldots \). The nilpotency class of \( G \) is denoted \( \text{Cl} (G) \) and \( \text{Cl} (G) = L \) iff \( G_{(L+1)} = E \) and \( G_{(L)} \neq E \), \( E \) the trivial subgroup containing the identity element only. If \( H_i, i = 1, \ldots, n, \) are subgroups of \( G \), then \( (H_1, \ldots, H_n) \) denotes the subgroup of \( G \) generated by \( \{(h_1, \ldots, h_n) \mid h_i \in H_i \} \). The well-known commutator identities

\[
(2.1) \quad (x, yz) = (x, z)(x, y)(x, y, z), \quad (xy, z) = (x, z)(x, z, y)(y, z),
\]

are used in the proof of Theorem 3.1.

For an extension \( W \) of \( A \) by \( B \) which is a semidirect product we assume both \( A \) and \( B \) are subgroups of \( W \).

If \( B = B_1 \times B_2 \times \cdots \times B_m \) is a direct product of finite \( p \)-groups, then \( B_{k,0} < \cdots < B_{k,L(k)} \) denotes an ascending central series of \( B_k \) contained in \( B \). Since \( B_{k,i}/B_{k,i-1} \) is a finite abelian \( p \)-group, \( B_{k,i} \) contains a minimal independent set of generators modulo \( B_{k,i-1} \). The elements of such a generating set are written \( \{b_{k,i,j}\}, i = 1, \ldots, r(k, i), r(k, i) \) being the cardinality of the generating set which is called the rank of the factor group.

**Definition 2.2.** Let \( p(k, i, j), 1 \leq j \leq r(k, i) \), denote the descending prime power orders of the cyclic groups in the decomposition of \( B_{k,i}/B_{k,i-1} \), \( k = 1, \ldots, m \) and \( i = 1, \ldots, L(k) \), for a specified ascending central series of length \( L(k) \) for \( B_k \). Define \( L = \max_{1 \leq k \leq m} L(k) \) and for \( L(k) < i \leq L \) set \( p(k, i, j) = 1 \) and \( r(k, i) = 0 \). Then we define

\[
\lambda_{ki} = \sum_{j=1}^{r(k, i)} (p(k, i, j) - 1), \quad 1 \leq k \leq m \text{ and } 1 \leq i \leq L,
\]

\[
d(k, t, s) = \prod_{j=t+1}^{s} p(k, j, 1), \quad 1 \leq k \leq m, 1 \leq s \leq L, \text{ and } 0 \leq t \leq s,
\]

and

\[
P_k(y_1, \ldots, y_s) = \sum_{t=1}^{s} d(k, t, s)y_t \quad \text{for } s = 1, 2, \ldots, L.
\]

The multivariable linear polynomials \( P_k(y_1, \ldots, y_s) \) have coefficients determined by the exponents of the factor groups \( B_{k,i}/B_{k,i-1} \) and the \( \lambda_{ki} \) are dependent upon the complete cycle structure of these factor groups.

3. **The upper bound.** In the theorems of this section \( B = B_1 \times \cdots \times B_m \) is a direct product of finite \( p \)-groups. The terms expressed are defined using arbitrary but specified ascending central series for the respective direct factors of \( B \) using Definition 2.2.

**Theorem 3.1.** Let \( W \) be a semidirect product of \( A \) by \( B \), \( A \) abelian of exponent \( p^{n+1} \). Then

\[
\text{Cl} (W) \leq \sum_{k=1}^{m} P_k(\lambda_{k1}, \ldots, \lambda_{kL(k)}) + n(p - 1)p^{-1} \max_{1 \leq k \leq m} d(k, 0, L(k)) + 1.
\]
Proof. Let
\[ c = \sum_{k=1}^{m} P_k(\lambda_{k1}, \ldots, \lambda_{kL(k)}) + n(p - 1)p^{-1} \max_{1 \leq k \leq m} d(k, 0, L(k)) + 1. \]

We will use induction on \( m \). The result for \( m = 1 \) is given in [5, Theorem 4.10]. So assume that the result holds for \( m - 1 \).

Assume \( \text{Cl} (W) > c \) and obtain a contradiction. Without loss of generality we assume \( \text{Cl} (W) = c + 1 \) and choose \( e \neq w \in W_{c+1} \). By [5, Corollary 3.4], \( c > L = \text{Cl} (B) \) and \( w = (x, b_0, b_1, \ldots, b_{c-q}) \) where \( b_0 \in B(q) \) and \( b_i \in B, 1 \leq i \leq c - q \) and \( 1 \leq q \leq L \). By the basic commutator identities, using the fact that \( A \) is abelian and \( W_{c+2} = E \), the map \( b_i \rightarrow (x, b_0, b_1, \ldots, b_{c-q}) \) is a homomorphism for each \( i, 0 \leq i \leq c - q \). So we may assume that \( b_i \in B_k \) for some \( k = k(i), 0 \leq i \leq c - q \). Since \( (B_k, B_j) = E \) for \( k \neq j \), [5, Lemma 4.6] allows us to assume that all the elements \( b_i \) from a given \( B_k \) follow each other. Let the number of entries from \( B_k \) be \( c_k \). Then \( c = \sum_{k=1}^{m} c_k \).

Without loss of generality we may assume that
\[ c_m > \sum_{k=1}^{m} P_k(\lambda_{m1}, \ldots, \lambda_{mL(m)}) + (n - t_m)(p - 1)p^{-1} d(m, 0, L(m)), \]
where \( t_m \) is minimal subject to this inequality holding. By [5, Theorem 4.10] applied to \( A \cdot B_m \), we may assume that \( (x, b_1, \ldots, b_{c_m}) \) has order dividing \( p^{t_m} \), where \( b_i \in B_m, 1 \leq i \leq m \).

If \( t_m = 0 \), then \( w = e \) and the contradiction is obtained. So assume that \( t_m \neq 0 \). Then, by the minimality of \( t_m \),
\[ c' = c - c_m \geq \sum_{k=1}^{m-1} P_k(\lambda_{k1}, \ldots, \lambda_{kL(k)}) \]
\[ + (t_m - 1)(p - 1)p^{-1} \max_{1 \leq k \leq m-1} d(k, 0, L(k)) + 1. \]

Let \( w' = (x', b_{c_m+1}, \ldots, b_c) \) where \( x' = (x, b_1, \ldots, b_{c_m}) \). By the induction hypothesis on \( m, w' = e \) since \( w' \in A \cdot (B_1 \times \cdots \times B_{m-1}) \). This gives the final contradiction.

The proof of the following theorem is an adaption of the proof of Theorem 5.12 of [4].

Theorem 3.2. Let \( W \) be a semidirect product of \( A \) by \( B = B_1 \times \cdots \times B_m \), \( A \) a nilpotent \( p \)-group of class \( c \) and \( B_k \) a finite \( p \)-group for each \( k = 1, \ldots, m \). Suppose \( A_0, A_1, \ldots, A_c \) is an ascending central series of \( A \). If \( A_j/A_{j-1} \) has exponent \( p^{n(j)} \) for \( 1 \leq j \leq c \), then
\[ \text{Cl} (W) \leq c \left( \sum_{k=1}^{m} P_k(\lambda_{k1}, \ldots, \lambda_{kL(k)}) \right) \]
\[ + \left( \sum_{j=1}^{c} (n(j) - 1)(p - 1)p^{-1} \max_{1 \leq k \leq m} d(k, 0, L(k)) \right) + c. \]
Proof. We proceed by induction on $c$. Theorem 3.1 is just the statement of this result for $c = 1$ so we let $c > 1$. Define

$$
t(j) = \sum_{k=1}^{m} P_{k}(\lambda_{k1}, \ldots, \lambda_{kL(k)}) + (n(j) - 1)(p - 1)p^{-1} \max_{1 \leq k \leq m} d(k, 0, L(k)) + 1,
$$

Now $A_1$ is a normal subgroup of $W$ and $W/A_1$ is a semidirect product of $A/A_1$ by $B$. By the induction hypothesis $\text{Cl}(W/A_1) \leq \sum_{j=2}^{c} t(j)$ since $(A_j/A_1)/(A_{j-1}/A_1)$ is isomorphic to $A_j/A_{j-1}$ for $2 \leq j \leq c$. Thus we have that $W(t) \subseteq A_1$ for $t = \sum_{j=2}^{c} t(j) + 1$. The result now follows from Theorem 3.1 since $A_1$ is contained in the centre of $A$ implies $(A_1, kW) \subseteq (A_1, kB)$.

References


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