ON THE FRACTIONAL PARTS OF $n/j, j = o(n)$

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Abstract. Dirichlet's result that if $J(n) = o(n)$ but $n^{1/2} = o(J(n))$, the numbers $n/j$ for $j = 1, \ldots, J(n)$ are nearly uniformly distributed modulo 1 (with error $\to 0$ as $n \to \infty$) is extended, $n^{1/2}$ being replaced by $n^a$ for any $a > 0$.

1. To illustrate the problem considered here (and the results): for large $n$, the real numbers $n/j$ for $j = 1, 2, \ldots, [n^{1/2}]$, reduced modulo 1, are nearly uniformly distributed. That is, for $t \in (0, 1)$, the fraction of those numbers $n/j$ that lie between $[n/j]$ and $[n/j] + t$ differs from $t$ by at most $\varepsilon(n)$, where $\varepsilon(n) \to 0$ as $n \to \infty$.

If $n^{1/2}$ is replaced by any function $J(n)$ satisfying $J(n) = o(n)$, but $n^{1/2} = o(J(n))$, the near-uniform distribution of those numbers is a result of Dirichlet (see [1, p. 327]), who showed also that the distribution is not uniform if $J(n) \neq o(n)$. This paper replaces Dirichlet's exponent $1/2$ by any $a > 0$.

Much the hardest part of the proof is due to A. Walfisz, who proved a lemma on the distribution of some of the numbers in question. In 1932 Walfisz applied his lemma to estimates of the number of lattice points in an ellipsoid [2]; in 1963 he gave some other applications [3]. But this application, which seems the most natural one, also seems never to have been done.

Theorem. If $J(n) = o(n)$ but some $n^a, a > 0$, is $o(J(n))$, then the fraction of the first $[J(n)]$ numbers $n/j (mod 1)$ which lie in an interval of length $t$ in $R/\mathbb{Z}$ differs from $t$ by at most $\varepsilon$, where $\varepsilon \to 0$ as $n \to \infty$.

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2. Walfisz's lemma concerns sums of the complex numbers $e(n/j) = \exp(2\pi in/j)$. The following specialization will suffice.

Lemma (Walfisz). Let $r$ be a positive integer, $w \in [0, 1]$, $R = 2^{-1}, R_1 = R(r + 1)$, $M$ between $n^{1/(r+2)}$ and $n^{2/(r+3)}$ (n need not be an integer). Then

$$\sum_{j=M}^{2M} e(n/(j + w)) = O(M^{-1/2}R^{-1/2}n^{1/R}\log n).$$

All we want of the concluding expression is that it is $o(M)$. Actually it is not, for the smallest $M$ allowed, viz. $n^{1/(r+2)}$; there the estimate is

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\[ O(M \log M) \] (and is, of course, worthless). We narrow the requirements by adding

\[(*) \quad M \geq n^{1/(r+2)}. \]

Still the different values of \( r \) give overlapping intervals from \( 2n^{1/2} \) down through all \( N^a \) in which the average value of \( M + 1 \) successive terms \( e(n/(j + w)) \) beginning at \( j = M \) is always (as a function of \( n \)) \( o(1) \). More precisely, for each \( r \), if \( n \) is large enough, each of those averages is less than \( \varepsilon \) (by Walfisz's proof and \( (*) \)).

\( 2n^{1/2} \) is not big enough (to adjoin Dirichlet's case). However, the \( o(1) \) conclusion extends all the way up \( o(n) \).

**Corollary.** For \( M(n) > n^{1/3} \), \( M(n) = o(n) \), the average of \( e(n/j) \) as \( j \) goes from \( M(n) \) to \( 2M(n) \) is \( o(1) \).

**Proof.** First, if we replace some \( n \) by \( n' = n/M(n) \), \( n' \) will still go to infinity with \( n \) and “\( o(1) \)” may be referred equally well to varying \( n \) or varying \( n' \). This still applies though, precisely, we introduce \( b(n) = 1 + [M(n)^2/n] \) and put \( n^* = n/b(n) \), so that \( n/j = n^*/(j/b(n)) \). Unless \( M(n) > n^{1/2} \), \( b(n) = 1 \) and we did nothing. Otherwise (with negligible error) we replace \( M \) by \( M^* = (n^*)^{1/2} \). Precisely, add at most \( b(n) - 1 \) terms to the sequence to make its length a multiple of \( b(n) \) (affecting the average by less than \( b(n)/M(n) = o(1) \)). The sequence of expressions \( e(n^*/(j/b(n))) \) decomposes into \( b(n) \) sequences in which denominators form progressions with difference \( 1 \); each has, with error \( o(1) \), the form in the lemma. So each has average \( o(1) \), and the average of those \( b(n) \) averages is still \( o(1) \).

3. We wish to apply Weyl's criterion, familiar in this form: a sequence \( (a_j) \) of complex numbers of modulus 1 is uniformly distributed if, for \( k = 1, 2, \ldots, \) the average of the first \( n \) \( k \)th powers \( a_j^k \) approaches 0 as \( n \to \infty \). We need the following form.

**Lemma (Weyl).** For each \( \varepsilon > 0 \) there exists \( N \) such that given a finite family of complex numbers of modulus 1, if for \( k < N \) the average of \( a_j^k \) has modulus less than \( 1/N \), then the fraction of the \( a_j \) which lie in an interval on the unit circle of length \( 2\pi \) is between \( t - \varepsilon \) and \( t + \varepsilon \).

Supposing this false, for some \( \varepsilon \) we should have a sequence of examples, \( N \to \infty \), missing by \( \varepsilon \) on certain intervals. For a subsequence, the intervals converge to a limit, and it is simple routine to patch together an infinite sequence \( (a_j) \) violating the criterion as previously stated, which is absurd.

The theorem follows. For, first, for \( M > n^a \) but \( o(n) \), the average of \( e(n/j) \) for \( j \) from \( M \) to \( 2M \) is small (for large \( n \)). Also \( M \) exceeds \( (2n)^{a/2} \), and the average of \( e(2n/j) \) for those \( j \) is small. This is true of \( e(kn/j) \), uniformly in \( k \), as long as \( k < n \), \( (kn)^{a/2} < n^a \). So the modified Weyl criterion tells us that those \( e(n/j) \) are uniformly distributed to within \( \varepsilon \) (\( \varepsilon \to 0 \) as \( n \to \infty \)). To within \( 2\varepsilon \), we can apply this to all \( j \) less than \( J(n) \), for \( J(n) > 2n^a \) but \( J(n) = o(n) \). Let \( M_0 = [2^{-i}J(n)] \), \( M_i = 2^iM_0 \) for \( i < s \); the \( s \) intervals \([M_i, 2M_i]\) reach from 1 to \( J(n) \), with negligible error and negligible overlap, and all have \( e(n/j) \) uniformly distributed (within \( \varepsilon \)).
REFERENCES


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