PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 60, October 1976

COHOMOLOGY OF ASSOCIATIVE TRIPLE SYSTEMS

RENATE CARLSSON

ABSTRACT. In the following a cohomology for associative triple systems is defined. The classical corollaries can be extended to the associative triples.

1. **Introduction**. One distinguishes two associative triple systems. The associative triples of the first kind or ternary rings were introduced by Lister [6]. Those of the second kind or Hestenes ternary rings were first investigated by Hestenes and as special alternative triples by Loos [7]. Typical examples for the latter are the $p \times q$ matrices in an associative algebra with an involution, the triple product given by $(x, y, z) \mapsto x\,{}^t\bar{y}z$, ${}^t\bar{y}$ the transposed-conjugate matrix.

In this note we define a canonic cohomology for associative triples. Cohomologies are well known for certain algebras and Lie triples [2]. The corollaries for associative triples are analogous to the classical results.

In the following K is a commutative and associative ring with unit 1, the modules taken unitary over K. If K is explicitly assumed as a field then the dimensions over K are finite.

2. **Definitions. The universal** (j-) **imbedding.** A K-module A with a trilinear inner composition $A \times A \times A \to A$, with $(x, y, z) \mapsto \langle xyz \rangle$, is an *associative triple system of the* 1*st kind* if

$$\langle\langle xyz\rangle uv\rangle = \langle x\langle yzu\rangle v\rangle = \langle xy\langle zuv\rangle\rangle$$

for $x, y, z, u, v \in A$. A is an *associative triple of the* 2*nd kind*, when

$$\langle\langle xyz\rangle uv\rangle = \langle x\langle uzy\rangle v\rangle = \langle xy\langle zuv\rangle\rangle.$$

A *trimodule* M over a triple system A is a K-module M together with 3 trilinear compositions $A \times A \times M \to M$, the products denoted $\langle xym\rangle$, $\langle xmy\rangle$, $\langle mxy\rangle$, where $m \in M$. The *split null extension* or *semidirect sum* $A + M$ of M over A is the K-module direct sum with

$$\langle (x + m_1)(y + m_2)(z + m_3)\rangle := \langle xyz\rangle + \langle m_1 yz\rangle + \langle xm_2 z\rangle + \langle xym_3\rangle$$

for $x, y, z \in A$, $m_i \in M$, $i = 1, 2, 3$. A trimodule M is *associative of the* 1*st kind* over A if the semidirect sum is an associative triple of the 1st kind, equally for the 2nd kind. In the following A always denotes an associative triple, and M an associative trimodule over A of the same kind if nothing

Received by the editors September 16, 1975.

AMS (MOS) subject classifications (1970). Primary 16A78, 18H15, 18H40; Secondary 08A25, 16A21, 16A28.

Key words and phrases. Cohomology groups of associative triple systems, universal (j-) imbedding, Whitehead lemmata, Wedderburn principal theorem, invariant submodules, trimodules.

Copyright © 1977, American Mathematical Society

other is stated, where associative means associative of the 1st or 2nd kind. For K-modules N, P let $N \oplus P$ denote the K-module direct sum.

Let B be an associative algebra, the multiplication designed by " $\circ$ ", and ι: $A \to B$ a K-module monomorphism. (B, ι) is an *imbedding* of an associative triple A of the 1st kind if $B = \iota A + \iota A \circ \iota A$ with $\iota x \circ \iota y \circ \iota z = \iota\langle xyz\rangle$, $x, y, z \in A$, and the imbedding is *direct* if the sum is direct [6]. (B, ι) is *universal* if for each homomorphism f: $A \to A'$ onto a triple A', and any imbedding (B', κ) of A' with monomorphism κ there exists a unique algebra homomorphism $\tilde{f}$: $B \to B'$ onto B' so that the diagram

$$\begin{tikzcd} B \arrow[r, "\tilde{f}"] & B' \\ A \arrow[u, hook, "\iota"] \arrow[r, "f"] & A' \arrow[u, hook, "\kappa"'] \end{tikzcd}$$

is commutative, $\hookrightarrow$ denoting a K-module monomorphism.

Let A be associative of the 2nd kind, j an involution of B with $\overline{\iota A} := j(\iota A)$, $\bar{b} := j(b)$, $b \in B$. (B, ι, j) is a *j-imbedding* of A if $B = \iota A + \overline{\iota A} + \iota A \circ \overline{\iota A} + \overline{\iota A} \circ \iota A$ with $\iota A \circ \iota A = \{0\}$ and $\iota x \circ \overline{\iota y} \circ \iota z = \iota\langle xyz\rangle$ for $x, y, z \in A$. The j-imbedding is *direct* if the above sum is direct. (B, ι, j) is *universal* if for any homomorphism f onto A', (B', κ, j') a j'-imbedding of A', there exists a unique homomorphism $\tilde{f}$: $B \to B'$ so that the above diagram is commutative and $j'\tilde{f} = \tilde{f}j$. The standard imbedding of A is a direct imbedding if A is associative of the 1st kind [6], and it contains a direct j-imbedding otherwise, as is easily noticed (cf. [7]). One immediately has

THEOREM 1. *Let A be an associative triple over K. Then there exists a universal imbedding $(U(A), \iota)$, respectively a universal j-imbedding $(U(A), \iota, j)$ of A. If $(V(A), \kappa)$, respectively $(V(A), \kappa, j')$, is a universal (j'-) imbedding too, then there is a unique isomorphism g of $U(A)$ onto $V(A)$ with $\kappa = g\iota$.*

PROOF. Let $T(A) := K \cdot 1 \oplus T^*(A)$ be the tensor algebra of the K-module A with $T^*(A) := \bigoplus(\otimes^n A)$, $n \in \mathbf{N}$. Then $T(A)$ is a universal enveloping associative algebra for A with unit 1. We construct $U(A)$ as a factor algebra: In case of the 2nd kind let $\bar{A}$ be a K-module isomorphic to A, the isomorphism given by $x \mapsto \bar{x}$. Let Q denote the ideal of $T^*(A \oplus \bar{A})$ generated by $x \otimes y$, $\bar{x} \otimes \bar{y}$, $x \otimes \bar{y} \otimes z - \langle xyz\rangle$, $\bar{x} \otimes y \otimes \bar{z} - \overline{\langle zyx\rangle}$ for $x, y, z \in A$. Set $U(A) := T^*(A \oplus \bar{A})/Q$ and ι: $x \mapsto x + Q$. Observing the elementary properties of the tensor product one notes the existence of an involution j of $U(A)$ with $j(x + Q) = \bar{x} + Q$. Thus $(U(A), \iota, j)$ is a universal j-imbedding of A. For the 1st kind take $T^*(A)$ modulo the ideal Q generated by the elements $x \otimes y \otimes z - \langle xyz\rangle$.

The proof of the second part of the statement is standard. $\square$

In the subsequent we may suppose that ι is the inclusion map. For an algebra C let $C^{[n]}$ denote the n-fold cartesian product. Let $U(A)$ design a universal (j-) imbedding of A. Obviously $U(A)$ is direct.

For a direct imbedding $B = A \oplus A \circ A$ let ε_1: $B \to A$, ε_2: $B \to A \circ A$ be

the canonic projections. If B is a direct j-imbedding let $\varepsilon_1, \varepsilon_2, \varepsilon_{-1}, \varepsilon_{-2}$ be its canonic projections onto A, $A \circ \bar{A}$, $\bar{A}$, $\bar{A} \circ A$. Set $B_k := \varepsilon_k(B)$, $k \in J := \{1, 2\}$, respectively $J := \{-2, -1, 1, 2\}$. The *formal multiplication* of the B-components B_k is obviously declared, e.g. for the 2nd kind $A \cdot (\bar{A} \circ A) := A$, $(A \circ \bar{A}) \cdot (A \circ \bar{A}) := A \circ \bar{A}$, $\bar{A} \cdot \bar{A} := \{0\}, \ldots$. Similar to the case of Lie triples [2] we will consider some homogeneous multilinear mappings of universal (j-) imbeddings. For $x := (x_1, \ldots, x_n) \in B^{[n]}$ with $x \in B_k$, $k = k(i)$, the B-component is $\Phi(x) := \prod B_{k(i)}$, $i = 1, \ldots, n$, the product formal. Let C be a subalgebra of B with $\varepsilon_k(C) \subset C$, and B' a (j'-) imbedding of A' with A' of the same kind as A. Then $f\colon C^{[n]} \to B'$ is *isovarying* if

$$\Phi(x) \subset B_k \Rightarrow f(x) \subset B'_k,$$

and *antivarying* if for $\Phi(x) \neq \{0\}$

$$\Phi(x) \subset B_k \Rightarrow f(x) \subset \bigoplus B'_l, \qquad l \neq k, l \in J.$$

Any f has a unique decomposition $f = f_1 + f_2$, f_1 isovarying and f_2 antivarying. Denote $\varepsilon\colon f \mapsto f_1$. Then $\varepsilon^2 := \varepsilon \cdot \varepsilon = \varepsilon$.

Let $B := A + M$, and let J be the ideal in $U(B)$ equal to

$$\{a \in B \circ M + M \circ B \mid a \circ A = A \circ a = \{0\}\},$$

respectively in the case of the 2nd kind

$$\{a \in B \circ \bar{M} + M \circ \bar{B} + \bar{B} \circ M + \bar{M} \circ B \mid a \circ (A + \bar{A}) = (A + \bar{A}) \circ a = \{0\}\}.$$

Then the *universal module imbedding* M_U is the ideal of $U(A + M)$ generated by M, respectively $M \oplus \bar{M}$, in the case of 2nd kind associativity, modulo J. The maps corresponding to ε and ε, j for the 2nd kind after the passages from $U(A + M)$ to $U(A)$ and M_U are denoted equally. M_U is an associative bimodule over $U(A)$ in the obvious way.

3. **The cohomology**. Let C^n be the linear space of the *n-cochains* of $U(A)$ for M_U [3], $n \in \mathbf{N}_0$. These are the n-linear mappings of $U(A)^{[n]}$ into M_U for $n \in \mathbf{N}$, while $C^0 = M_U$. The coboundary operator is defined by

$$\delta f(a_1) = a_1 \cdot f - f \cdot a_1, \quad \text{if } f \in C^0,$$

$$\begin{aligned}\delta f(a_1, \ldots, a_{n+1}) = {} & a_1 \cdot f(a_2, \ldots, a_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_i \circ a_{i+1}, \ldots, a_{n+1}) \\ & + (-1)^{n+1} f(a_1, \ldots, a_n) \cdot a_{n+1} \\ & \qquad \text{for } f \in C^n, n \neq 0, a_i \in U(A).\end{aligned}$$

Z^n, B^n denote the K-modules of the n-cocycles and the n-coboundaries where $B^0 := \{0\}$. $H^n(U(A), M_U) := Z^n / B^n$ is the n-dimensional cohomology group. If $x = (x_1, \ldots, x_n) \in U(A)^{[n]}$, $n \neq 0$, $U(A)$ a j-imbedding, set $j(x) := (j(x_n), \ldots, j(x_1))$. For $f \in C^0$ extend ε by $\varepsilon(f) := \varepsilon_2 f$ if A is of the 1st kind, and if A is of the 2nd kind let $\varepsilon(f) := (\varepsilon_2 + \varepsilon_{-2}) f$, $fj := f$. Then the K-module

of the *n-cochains of the associative triple system A for M* is

$$C^n(A, M) := \{f \in C^n | \varepsilon(f) = f\}$$

for $n \in \mathbf{N}_0$ if A and M are associative of the 1st kind. In case of the 2nd kind define

$$C^n(A, M) := \{f \in C^n | \varepsilon(f) = f, jfj = (-1)^{n(n+1)/2+1} f\}$$

for $n \in \mathbf{N}_0$ if $\frac{1}{2} \in K$. If $2 \cdot K = \{0\}$ modify this definition by $C^0(A, M) := \{f \in C^0 | \varepsilon(f) = f, jf = f + z$ with $z \in M_U$, $a \cdot z = z \cdot a$ for all $a \in U(A)\}$.

The *coboundary operator* ∂ is the restriction of δ from C^n onto $C^n(A, M)$. The modules $C^n(A, M)$ together with ∂ define a *cochain complex*. $Z^n(A, M)$ is the submodule of *n-cocycles* of $C^n(A, M)$, $B^n(A, M)$ that of *n-coboundaries*. Let $H^n(A, M)$ be the *n-dimensional cohomology group* with $H^n(A, M) := Z^n(A, M)/B^n(A, M)$. Set $Z^n_+ := Z^n \cap C^n(A, M)$, and let $B^n_+ := B^n \cap C^n(A, M)$. We have

LEMMA 1. *Let A be an associative triple system, M an associative trimodule over A, and* $\frac{1}{2} \in K$ *in case of the 2nd kind. Then*

$$H^n(A, M) = Z^n_+/B^n_+, \quad \text{and} \quad H^n(U(A), M_U) \cong H^n(A, M) \oplus H(n),$$

$H(n)$ a submodule of $H^n(U(A), M_U)$, $n \in \mathbf{N}_0$.

PROOF. Let A and M be associative of the 2nd kind, and $\frac{1}{2} \in K$. Then for $f \in C^n$ we have $f = \varepsilon(f) + (f - \varepsilon(f))$ and $f = \frac{1}{2} \cdot ((f + jfj) + (f - jfj))$. These are the decompositions of f into eigenfunctions for the endomorphisms ε and $\tilde{j}$: $f \mapsto jfj$. An easy verification shows that ε and $\tilde{j}$ commute. Thus there is a weight space decomposition $C^n = \bigoplus C^n_{\alpha,\beta}$, α the root of ε, β the root of $\tilde{j}$ for $C^n_{\alpha,\beta}$. Now $C^n(A, M) = C^n_{1,(-1)^{n(n+1)/2+1}}$. Since δ: $C^{n-1}_{\alpha,\beta} \to C^n_{\alpha,\beta'}$, $\beta' = \beta \cdot (-1)^n$, we get $B^n(A, M) = B^n_+$. The rest is obvious. For the case of the 1st kind one proceeds similarly and even more immediately. □

The associative triple A is *separable over the field K*, if for every base field extension $\tilde{K}$ of K the base field extension $\tilde{A}$ of A is semisimple. Note that as in the binary case the radical of a ternary algebra A coincides with the radical of the ternary ring A when $\frac{1}{2} \in K$ [8]. We receive

COROLLARY 1. *Let M be an associative trimodule of the 1st or 2nd kind over A, and A separable over a field K,* $\mathrm{char}(K) \neq 2$. *Then for all* $n \in \mathbf{N}, H^n(A, M) = \{0\}$.

PROOF. Because A is semisimple the standard imbedding of A is semisimple too by [6], respectively [7]. Any semisimple imbedding, respectively j-imbedding, is isomorphic to $U(A)$. The isomorphism follows as the corresponding one in case of Lie triples [4]. Thus $U(A)$ is separable and

$$H^n(U(A), M_U) = \{0\} \quad \text{for all } n \in \mathbf{N}. \qquad \square$$

Corollary 1 includes the statements corresponding to the two *Whitehead lemmas*.

A linear mapping D: $A \to M$ is a *derivation* of A in M if

$$D(\langle xyz \rangle) = \langle D(x)yz \rangle + \langle xD(y)z \rangle + \langle xyD(z) \rangle, \qquad x, y, z \in A.$$

Inner derivations are those generated by the triple multiplications in the semidirect sum $A + M$.

If $\mathrm{Der}(A, M)$ is the K-module of derivations of A in M, then we have $Z^1(A, M) \cong \mathrm{Der}(A, M)$, the elements of $B^1(A, M)$ related to inner derivations of A in M. As isomorphism of $Z^1(A, M)$ onto $\mathrm{Der}(A, M)$ we may choose the restriction of the elements $f \in Z^1(A, M)$ from $U(A)$ onto A. It follows immediately that in case of the 2nd kind, $\mathrm{char}(K) = 2$ and $n = 1$, Corollary 1 is valid too. Hence

COROLLARY 2. *Let A and M be associative of the same kind, A separable over the field K, and* $\mathrm{char}(K) \neq 2$ *in case of the* 1*st kind. Then every derivation of A in M is inner.* □

We may remark that in case of $\mathrm{char}(K) = 2$ the trivial 1st kind triple $A := \{0, 1\}$ with $\langle 111 \rangle = 1$ is separable, while the radical of $U(A)$ is $\{0, 1 + 1 \circ 1\} \neq \{0\}$.

Now let C denote a binary algebra, N a C-bimodule. A *semiautomorphism* f of N is an algebra automorphism of the semidirect sum $C + N$ with $f(A) \subset A, f(N) \subset N$. It holds:

LEMMA 2. *Let G be a finite group of semiautomorphisms of a bimodule N over an algebra C, n the order of G, and $1/n \in K$. If $N = Q \oplus P$, Q and P are C-submodules with $G(Q) = Q$, then there exists a C-submodule R of N with $G(R) = R$ and $N = Q \oplus R$.* □

The proof follows considering the semidirect sum $C + N$ and going on along the lines of [10].

Irreducible trimodules, complete reducibility, triple (trimodule) homomorphisms are defined as obvious. If A is semisimple, Z the center of $U(A)$, and $Z \otimes Z$ semisimple, A is *c-semisimple*. Denote $W := U(A) \otimes U(A)^{\mathrm{op}}$, $U(A)^{\mathrm{op}}$ the opposite algebra for $U(A)$. For c-semisimple A, W is semisimple. For a direct imbedding B let σ be the natural algebra automorphism with $\sigma(x) = -x$ and fixing $x \circ y$ when $x, y \in A$.

PROPOSITION 1. *Let M be an associative trimodule over A, K a field with* $\mathrm{char}(K) \neq 2$, *and A c-semisimple. Then M is completely reducible over K.*

PROOF. M_U is an associative left W-module by $x \otimes y \cdot m = x \circ m \circ y$ with $x, y \in U(A)$, $m \in M_U$. Then $M_U = e \circ M_U \circ e \oplus M_0$, e the unit of $U(A)$, M_0 the maximal zero W-submodule, is a decomposition into W-submodules. Set $M := \underline{e} \circ M_U \circ \underline{e}$. It may be noted that for the 2nd kind $e = e_1 + e_2$ with $e_1 \in A \circ \bar{A}$, $e_2 \in \bar{A} \circ A$, so that $\varepsilon_1(M) = e_1 \circ M_U \circ e_2$. Thus $M = \varepsilon_1(M) \oplus \varepsilon_1(M_0)$ is a decomposition of M into subtrimodules. Consider now $\varepsilon_1(M_0)$ as a left- and a right-module over the semisimple algebra $A \circ A$ over which it decomposes into irreducible submodules [6], and equally for $A \circ \bar{A} \oplus \bar{A} \circ A$ in case of the 2nd kind. Hence $\varepsilon_1(M_0)$ is completely reducible over A.

Set $G := \{\mathrm{Id}, \sigma\}$ for the 1st kind. Then $G(M) \subset M$. If $M \neq \{0\}$, M decomposes by Lemma 2 or [5, p. 117] into minimal G-invariant submodules $P_k \neq \{0\}$ over W, $k = 1, \ldots, t$. Therefore $\varepsilon_1(P_k) \subset P_k$. Hence $N_k := \varepsilon_1(P_k)$ is an irreducible subtrimodule. Thus M is a completely reducible A-trimodule. □

The proposition says equivalently that under its conditions on A any short exact sequence of associative A-trimodules of the same kind, the connecting mappings being trimodule homomorphisms, splits by a trimodule homomorphism.

An *extension of the associative triple A by M* is a short exact sequence of associative triples M, E, A of the same kind $\{0\} \to M \to^e E \to^p A \to \{0\}$ with e, p triple homomorphisms. M is in a canonic way a trimodule over A. Let ${}^{(1)}M := \langle MMA \rangle + \langle MAM \rangle + \langle AMM \rangle$. The extension is *singular* if ${}^{(1)}M = \langle MMM \rangle = \{0\}$. The extension is *split* or *trivial* if there exists a triple homomorphism q: $A \to E$, with $pq = \mathrm{Id}_A$. The extension $\{0\} \to M \to^f E' \to^q A \to \{0\}$ is *equivalent* to the first if there is a triple homomorphism k: $E \to E'$ so that the diagram

$$\begin{tikzcd}
 & & E \arrow[dd, "k"] \arrow[dr, "p"] & & \\
\{0\} \arrow[r] & M \arrow[ur, "e"] \arrow[dr, "f"'] & & A \arrow[r] & \{0\} \\
 & & E' \arrow[ur, "q"'] & &
\end{tikzcd}$$

is commutative. Then k is an isomorphism. Let J' be the ideal of $U(E)$ defined as J, substituting $e(M)$, E for M and $A + M$. Set $U(E)' := U(E)/J'$. Design the maps induced by ε and j on $U(E)'$ equally. For a singular extension there is a commutative diagram:

$$\begin{tikzcd}
\{0\} \arrow[r] & M \arrow[r, "e"] \arrow[d, hook] & E \arrow[r, "p"] \arrow[d, hook] & A \arrow[r] \arrow[d, hook] & \{0\} \\
\{0\} \arrow[r] & M_U \arrow[r, "e'"] & U(E)' \arrow[r, "p'"] & U(A) \arrow[r] & \{0\}
\end{tikzcd}$$

e', p' algebra homomorphisms invariant relative to ε and respectively for j. Note that the second row is also exact (cf. [2, pp. 158–159]).

A *factor set* f of A in M is, as usual, a trilinear mapping f: $A \times A \times A \to M$ so that the K-module direct sum $A + M$ with $[xyz] := \langle xyz \rangle + f(x, y, z)$, ${}^{(1)}M := \{0\}$, $\langle MMM \rangle := \{0\}$, and the trimodule structure of M, is an associative triple of the same kind. f is *trivial* if there is s: $A \to M$, s K-linear with

$$f(x, y, z) := s(\langle xyz \rangle) - \langle s(x)yz \rangle - \langle xs(y)z \rangle - \langle xys(z) \rangle.$$

Let A be K-projective in the usual sense. Then any extension of the triple A is split for the K-module A. Similarly as for algebras, the classes of equivalent singular extensions of A by M constitute a K-module isomorphic to the K-module of factor sets, the trivial factor sets corresponding to the classes of the split extensions. One then obtains, arguing as [2, pp. 159–160] for Lie triples,

THEOREM 2. *If A and the trimodule M over A are associative of the same kind, A K-projective, and $\frac{1}{2} \in K$ in case of the 2nd kind, then there is a module isomorphism $H^2(A, M) \cong E(A, M)/E_t(A, M)$, $E(A, M)$ the K-module of the classes of equivalent singular extensions of A with kernel M, $E_t(A, M)$ the submodule of the classes of split extensions.* $\square$

If $A^{(1)} := \langle AAA \rangle$, $A^{(n+1)} = (A^{(n)})^{(1)}$, $n \in \mathbf{N}$, then A is *solvable* when $A^{(m)} = \{0\}$ for some m. Set for an ideal I in A, ${}^{(n+1)}I := {}^{(1)}({}^{(n)}I)$. I is *S-solvable* if for one m, ${}^{(m)}I = \{0\}$. Because $R := \mathrm{Rad}(A) \subset \mathrm{Rad}(U(A))$, R is solvable if K is a field. We have for arbitrary K

LEMMA 3. *Let I be an ideal of A. Then*:

(a) *I is solvable exactly if it is S-solvable.*

(b) *When I is solvable, $I \neq \{0\}$, then ${}^{(1)}I \subsetneqq I$.*

PROOF. (a) From $B^{(1)} \subset {}^{(1)}B$ we get $B^{(n)} \subset {}^{(n)}B$ for any $n \in \mathbf{N}$. Further one verifies ${}^{(3)}B \subset B^{(1)}$. Thus ${}^{(3n)}B \subset B^{(n)}$. Hence we get (a). (a) $\Rightarrow$ (b). $\square$

The proof of the *Wedderburn principal theorem* reduces by Lemma 3 to the singular case. For associative triples of the 2nd kind the statement is implied by [1]. On behalf of Corollary 1 it follows:

COROLLARY 3. *Let A be a finite dimensional associative triple system over a field K, R the radical, A/R separable, and* $\mathrm{char}(K) \neq 2$. *Then A decomposes $A = B \oplus R$, B a subtriple of A, $B \cong A/R$.* $\square$

REFERENCES

1. R. Carlsson, *Der Wedderburnsche Hauptsatz für alternative Paare* (to appear).

2. B. Harris, *Cohomology of Lie triple systems and Lie algebras with involution*, Trans. Amer. Math. Soc. **98** (1961), 148–162. MR **22** #11068.

3. G. Hochschild, *On the cohomology groups of an associative algebra*, Ann. of Math. (2) **46** (1945), 58–67. MR **6**, 114.

4. N. Jacobson, *General representation theory of Jordan algebras*, Trans. Amer. Math. Soc. **70** (1951), 509–530. MR **12**, 797.

5. ——, *Structure of rings*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 37, Amer. Math. Soc., Providence, R. I., 1964. MR **36** #5158.

6. W. G. Lister, *Ternary rings*, Trans. Amer. Math. Soc. **154** (1971), 37–55. MR **42** #7716.

7. O. Loos, *Assoziative Tripelsysteme*, Manuscripta Math. **7** (1972), 103–112. MR **46** #3581.

8. H. C. Myung, *A characterization of the Jacobson radical in ternary algebras*, Proc. Amer. Math. Soc. **38** (1973), 228–234. MR **49** #363.

9. P. Seibt, *Cohomology of algebras and triple systems*, Comm. Algebra **3** (1975), 1097–1120.

10. E. J. Taft, *Cleft algebras with operator groups*, Portugal. Math. **20** (1961), 195–198. MR **25** #2103.

UNIVERSITÄT HAMBURG, MATHEMATISCHES SEMINAR, BUNDESSTRASSE 55,2 HAMBURG 13, FEDERAL REPUBLIC OF GERMANY