PULL-BACKS OF FORMATION PROJECTORS

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Abstract. Pull-backs for the projectors associated with a formation of finite solvable groups are constructed. An application is given to the construction of certain interesting formations.

1. Introduction. In this note, we show that "pull-backs" exist for the SF-projectors associated with a saturated formation \( \mathcal{F} \) of finite solvable groups. This result enables us to construct in a uniform manner several interesting classes of formations.

Our notation and terminology are fairly standard, and can be understood by referring to [4] or [6]. The latter source contains the basic material on formations needed here. Note that we use the more current term \( \mathcal{F} \)-projector for what were formerly called \( \mathcal{F} \)-covering subgroups.

2. Preliminaries. The constructions of §§3 and 4 depend on an important theorem of B. Huppert [7].

Theorem 2.1. Suppose that \( \mathcal{F} \) is a saturated formation, and that \( F \) is an \( \mathcal{F} \)-projector of a solvable group \( G \). Then for all normal subgroups \( M \) and \( N \) of \( G \),

\[
F \cap MN = (F \cap M)(F \cap N).
\]

We shall need the next result in the proof of Theorem 4.2.

Theorem 2.2 [5]. Suppose that \( C \) is a subgroup of a solvable group \( G \). If \( C \) centralizes each \( p \)-chief factor of \( G/\Phi(G) \), then \( C \) centralizes each \( p \)-chief factor of \( G \) below \( \Phi(G) \).

The following proposition is common folklore, although not perhaps in this form. We denote by \( \mathcal{F}_1 \mathcal{F}_2 \) the class of all groups which are extensions of groups in \( \mathcal{F}_1 \) by groups in \( \mathcal{F}_2 \).

Proposition 2.3. Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be saturated formations, and suppose that the order of each group in \( \mathcal{F}_1 \) is coprime to the order of each group in \( \mathcal{F}_2 \). Then \( \mathcal{F}_1 \mathcal{F}_2 \) is a formation. If \( F_1 \) is an \( \mathcal{F}_1 \)-projector of \( G \), and \( F_2 \) is an \( \mathcal{F}_2 \)-projector of \( N_G(F_1) \), then \( F_1 F_2 \) is an \( \mathcal{F}_1 \mathcal{F}_2 \)-projector of \( G \).

Proof. The class \( \mathcal{F}_1 \mathcal{F}_2 \) is clearly quotient closed. Suppose that \( G/M \) and \( G/N \) belong to \( \mathcal{F}_1 \mathcal{F}_2 \). Then for some normal subgroups \( R, S \), we have
Choose a Hall $\pi$-subgroup $H$ of $G$, where $\pi$ is the set of primes dividing $|R/M|$ or $|S/N|$. Then by the hypothesis on $\mathcal{F}_1$ and $\mathcal{F}_2$, $H \leq R \cap S$, and $HM = R, HN = S$. Thus $H/H \cap M \cong R/M$ and $H/H \cap N \cong S/N$. Since $R \cap S/M \cap N$ is a $\pi$-group, $H(M \cap N) = R \cap S$, and so

$$R \cap S/M \cap N \cong H/H \cap M \cap N \in \mathcal{F}_1.$$ 

Now since $G/R \cap S \in \mathcal{F}_2$, we have $G/M \cap N \in \mathcal{F}_1 \mathcal{F}_2$, as desired.

We consider the second conclusion. Certainly $F_1 F_2 \in \mathcal{F}_1 \mathcal{F}_2$. Now suppose that $F_1 F_2 \leq V$, and $V/N \in \mathcal{F}_1 \mathcal{F}_2$. We must verify that $F_1 F_2 N = V$. Since $V/N \in \mathcal{F}_1 \mathcal{F}_2$, there is a normal subgroup $K$ of $V$ so that $V/K \in \mathcal{F}_2$ and $K/N \in \mathcal{F}_1$. Since $|V/K|$ is coprime to $|F_1|$, $F_1 \leq K$, and therefore $F_1$ is an $\mathcal{F}_1$-projector of $K$. Thus $K = F_1 N$. Since any two $V$-conjugates of $F_1$ are necessarily $K$-conjugate, $V = K \cdot N_\mathcal{F}(F_1)$. Therefore $V/K \cong N_\mathcal{F}(F_1) \cap K$. This last quotient belongs to $\mathcal{F}_2$, and $N_\mathcal{F}(F_1)$ contains $F_2$, an $\mathcal{F}_2$-projector of $N(F_1)$. It follows that $F_2(N_\mathcal{F}(F_1) \cap K) = N_\mathcal{F}(F_1)$; now $V = KN_\mathcal{F}(F_1) = KF_2 = NF_1 F_2$, completing the proof.

3. Pull-backs for $\mathcal{F}$-projectors. We show now that pull-backs exist for the projectors associated with a saturated formation $\mathcal{F}$.

**Theorem 3.1.** Let $\mathcal{F}$ be a saturated formation. Let $G$ be a solvable group with normal subgroups $M$ and $N$, and suppose that $M \cap N = 1$. If an $\mathcal{F}$-projector of $G/M$ and an $\mathcal{F}$-projector of $G/N$ have the same image in $G/MN$, then they are both homomorphic images of some $\mathcal{F}$-projector of $G$.

**Proof.** Since pre-images of $\mathcal{F}$-projectors contain $\mathcal{F}$-projectors, we may suppose that there are $\mathcal{F}$-projectors $F_1, F_2$ of $G$, such that $F_1 MN = F_2 MN$. We must find an $\mathcal{F}$-projector $F$ so that $FM = F_1 M$ and $FN = F_2 N$. It is routine to see that if $G$ is a minimal counterexample to the theorem, $G = F_1 MN$ and $M$ and $N$ are minimal normal subgroups in $G$, and therefore abelian. A nontrivial intersection $F_1 \cap M$ or $F_1 \cap N$ would be normal in $G$; therefore $F_1$ must either contain or have trivial intersection with each of $M$ and $N$. Suppose that $M \leq F_1$. Then $G/N = F_1 N/N \in \mathcal{F}$, so that $F_2 N = G$. Thus we may take $F = F_1$ and be finished, in this case. By symmetry, we may suppose that $M \cap F_1 = N \cap F_1 = 1$. Now by the theorem of B. Huppert given above, $F_1 \cap MN = 1$, and we may suppose that similar results hold for $F_2$. Now let $F = F_1 M \cap F_2 N$. Then $|F| = |F_1 M| |F_2 M|/|G| = |F_1|$. Now $MF = M(F_1 M \cap F_2 N) = F_1 M \cap G$ by the modular law. This shows that $F$ and $F_1$ are both complements to $M$ in $MF$, and are therefore conjugate. In particular, $F$ is an $\mathcal{F}$-projector of $G$. Since we can show in identical fashion that $NF = NF_2$, our proof is complete.

4. Constructing formations by comparing projectors. It is easy to see that the class of groups for which $\mathcal{F}_1$-projectors belong to $\mathcal{F}_2$ is a formation; $4$-groups, and groups with Abelian Carter subgroups are two familiar formations of this type. As an application of the results above, we offer a more delicate construction.

**Theorem 4.1.** Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are saturated formations; then the class
of groups in which $F_1$-projectors are subgroups of $F_2$-projectors is a formation. In particular, the class of groups for which $F_1$- and $F_2$-projectors coincide is a formation.

Proof. The second statement is a direct consequence of the first. Since projectors are carried onto projectors by epimorphisms, these classes are quotient closed. Now suppose that $M$ and $N$ are normal in $G$, $M \cap N = 1$, and in both $G/M$ and $G/N$, $F$-projectors are subgroups of $F_1$-projectors. Then for an $F_2$-projector $F_2$ of $G$, $F_2 M/M$ and $F_2 N/N$ are $F_2$-projectors of $G/M$ and $G/N$. Furthermore, by lifting back from $G/MN$ into $G/M$ and $G/N$, and then applying Theorem 3.1, we may assume that for some $F_1$-projector $F_1$, we have $F_1 M \leq F_2 M$ and $F_1 N \leq F_2 N$. Applying 2.1, we see that

$$F_1 = F_1(M \cap N) = F_1 M \cap F_1 N \leq F_2 M \cap F_2 M = F_2(M \cap N) = F_2;$$

our proof is complete.

Several important formations arise in the manner of Theorem 4.1. For instance, groups in which Carter subgroups (i.e. nilpotent projectors) are Hall subgroups have arisen in investigations by R. Carter, J. G. Thompson and others, of splitting and nilpotency length (see [1], [8]; also [6, p. 743]). J. B. Derr [2] and A. Fattahi [3] have studied a generalization of Sylow tower groups of this type, in which the normalizers of Sylow $p$-subgroups (projectors for $p$-by-$p'$ groups) are Hall $\pi$-subgroups (projectors for $\pi$-groups).

The formations we have constructed are usually not saturated. For example, let $G$ be the non-nilpotent extension of the quaternion group $Q_8$ by a cyclic group $C_3$. Then in $G/\Phi(G)$, the Carter subgroup is a Sylow subgroup, but this is false in $G$. For many classes which are more or less arithmetically defined, we can prove saturation.

Theorem 4.2. Let $\pi_1, \ldots, \pi_k$ be disjoint sets of primes, and set $\pi = \bigcup \pi_i$. Let $S_{\pi}$ denote the formation of solvable $\pi$-groups. Then the class of groups $S$ in which an $S_{\pi_1} \cdots S_{\pi_k}$-projector forms a saturated formation.

Proof. In view of 4.1, only saturation needs proof. Suppose that $M$ is a minimal normal of $G$, with $M \leq \Phi(G)$. Then $M$ is a $p$-group for some prime $p$. If $p \not\in \pi$, then $G/M$ and $G$ have isomorphic Hall $\pi$-subgroups, so that $G \in \mathcal{F}$ if $G/M \in \mathcal{F}$. Now suppose that $p \in \pi_i$. Then a Hall $\pi$-subgroup of $G$ contains $M$, so we need to see that an $S_{\pi_1} \cdots S_{\pi_k}$-projector does also. In view of 2.3, this is no difficulty if $p \in \pi_i$. If $i > 1$, the hypothesis on $G/M$ and 2.3 show that for $r \in \pi_1 \cup \cdots \cup \pi_{i-1}$, a Sylow $r$-subgroup centralizes each $p$-factor of $G/M$. Now by 2.2, this is also true in $G$, so $M$ is centralized by a Hall $\pi_1 \cup \cdots \cup \pi_{i-1}$-subgroup of $G$, and therefore $M$ is contained in an $S_{\pi_1} \cdots S_{\pi_k}$-projector, as desired. This completes the proof of the theorem.

Bibliography


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