A REMARK ON THE STRONG LAW OF LARGE NUMBERS

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Abstract. Let $X_1, X_2, \ldots$ be mutually independent random variables such that $E(X_n) = 0$ and $E(X_n^2) = \sigma_n^2 = 1$ for all $n = 1, 2, \ldots$. For each $n = 1, 2, \ldots$, let $S_n = \sum_{j=1}^n X_j$; then, by the Kolmogorov criterion for mutually independent random variables, $S_n/n^{1/2+\alpha} \to 0$ almost surely as $n \to \infty$ for any positive constant $\alpha$. A deeper understanding of this theorem will be facilitated if we know the order of magnitude of $E(N_\infty(a, \epsilon))$ as $\epsilon \to 0^+$, where $N_\infty(a, \epsilon)$ is the integer-valued random variable defined by $N_\infty(a, \epsilon) = \sum_{n=1}^\infty X_n e^{(1/2+\alpha)e^{-1/2+\epsilon}}$. The present note does the work for a wide class of random variables by using Esseen's theorem and Katz-Petrov's theorem.

Let $X_1, X_2, \ldots$ be mutually independent random variables such that $E(X_n) = 0$ and $E(X_n^2) = \sigma_n^2 = 1$ for all $n = 1, 2, \ldots$. For each $n = 1, 2, \ldots$, let $S_n = \sum_{j=1}^n X_j$; then, by the Kolmogorov criterion for mutually independent random variables, $S_n/n^{1/2+\alpha} \to 0$ almost surely as $n \to \infty$ for any positive constant $\alpha$. For any positive constants $\alpha$ and $\epsilon$, let $A_n(a, \epsilon) = X_n e^{(1/2+\alpha)e^{-1/2+\epsilon}}$ for all $n = 1, 2, \ldots$ and let $N_\infty(a, \epsilon) = \sum_{n=1}^\infty A_n(a, \epsilon)$. Then, for a deeper understanding of the theorem above, we will study the order of magnitude of $E(N_\infty(a, \epsilon))$ as $\epsilon \to 0^+$, and this is just the main purpose of this note. We start with the following useful lemmas.

Lemma 1. Suppose that $X_1, X_2, \ldots$ are mutually independent, standard normal random variables, and $\alpha, \epsilon$ are two positive constants. Then, we have

$$C_\alpha \epsilon^{-1/\alpha} - 1 \leq E(N_\infty(a, \epsilon)) \leq C_\alpha \epsilon^{-1/\alpha},$$

where $C_\alpha = \pi^{-1/2} 2^{1/2+\alpha} \Gamma(1/2 + 1/2\alpha)$.

Proof. For any positive integer $m$, let $N_m(a, \epsilon) = \sum_{n=1}^m A_n(a, \epsilon)$; then $1 + E(N_m(a, \epsilon)) = 2 \sum_{n=0}^m \Phi(-en^\alpha)$, where $\Phi(x)$ is the distribution function of the standard normal random variable. By the Euler-Maclaurin sum formula (see [1, pp. 124-125]),

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\[ 2 \sum_{n=0}^{m} \Phi(-\varepsilon n^a) = \frac{1}{2} + \Phi(-\varepsilon m^a) + 2 \int_{0}^{m} \Phi(-\varepsilon x^a) \, dx - 2 \int_{0}^{m} R(x) \, d\Phi(-\varepsilon x^a), \]

where \( R(x) = [x] - x + 1/2 \) and \([x]\) denotes the greatest integer not exceeding \( x \). Letting \( m \to \infty \) and by the monotone convergence theorem, we get

\[ \frac{1}{2} + E(N_{\infty}(\alpha, \varepsilon)) = 2 \int_{0}^{\infty} \Phi(-\varepsilon x^a) \, dx - 2 \int_{0}^{\infty} R(x) \, d\Phi(-\varepsilon x^a), \]

hence

\[ 2 \int_{0}^{\infty} \Phi(-\varepsilon x^a) \, dx + \int_{0}^{\infty} d\Phi(-\varepsilon x^a) \leq \frac{1}{2} + E(N_{\infty}(\alpha, \varepsilon)) \leq 2 \int_{0}^{\infty} \Phi(-\varepsilon x^a) \, dx - \int_{0}^{\infty} d\Phi(-\varepsilon x^a). \]

Therefore,

\[ c_{\alpha} \varepsilon^{-1/a} - 1 \leq E(N_{\infty}(\alpha, \varepsilon)) \leq c_{\alpha} \varepsilon^{-1/a}. \]

For Lemmas 2 and 3, let \( X_1, X_2, \ldots \) be mutually independent random variables such that \( E(X_n^2) = 0, E(X_n^2) = \sigma_n^2 < \infty \) (not all of \( \sigma_n^2 \)'s are zero), \( S_n = \sum_{j=1}^{n} X_j, B_n^2 = \sum_{j=1}^{n} \sigma_j^2, \Phi_n(x) = P(S_n/B_n \leq x) \) for all \( n = 1, 2, \ldots, \) and let \( G \) be the class of nonnegative functions \( g(x) \) satisfying the following conditions: (i) \( g(x) \) is nondecreasing on the interval \((0, \infty)\), is even on \((-\infty, 0)\) and \( g(x) \to \infty \) as \( x \to 0^+ \); (ii) the function \( x/g(x) \) does not decrease on \((0, \infty)\).

**Lemma 2 (Petrov).** If there exists a function \( g \) in \( G \) such that \( E(X_n^2 g(X_n)) < \infty \) for all \( n = 1, 2, \ldots \), then there exists an absolute constant \( C^* \) such that

\[ (2) \quad \Delta(n) = \sup_{-\infty < x < \infty} |\Phi_n(x) - \Phi(x)| \leq \frac{C^*}{B_n^2 g(B_n)} \sum_{j=1}^{n} E(X^2 g(X_j)). \]

**Proof.** See [4, pp. 242–244].

**Lemma 3 (Esseen).** If \( \Delta(n) \leq \frac{1}{2} \) for all \( n > n_0 \), then there exists an absolute constant \( C^*_1 \) such that

\[ (3) \quad |\Phi_n(x) - \Phi(x)| \leq \min \left\{ \Delta(n), C^*_2 \cdot \frac{\Delta(n) \log(1/\Delta(n))}{1 + x^2} \right\}, \]

for all \( n > n_0 \) and all values of \( x \).

**Proof.** See [2, pp. 68–70].

Now we are in the position to state and prove our main theorem and corollaries.
Theorem 1. Suppose that $X_1, X_2, \ldots$ are mutually independent random variables such that $E(X_n) = 0$ and $E(X_n^2) = \sigma_n^2 = 1$ for all $n = 1, 2, \ldots$. If there exists a function $g$ in $G$ such that 
(i) $\limsup_{n \to \infty} n^{-1} \sum_{j=1}^{n} E(X_j^2 g(X_j)) < \infty$, 
(ii) $\sum_{n=1}^{\infty} (\log n / n^{2\alpha} g(\sqrt{n})) < \infty$ for some constant $\alpha$ ($0 < \alpha \leq \frac{1}{2}$), then, we have

$$\lim_{\varepsilon \to 0^+} \varepsilon^{1/\alpha} E(N_{\infty}(\alpha, \varepsilon)) = C_\alpha.$$ 

Proof. The proof will be given by the following two steps.

Step 1. Assume that $X_1, X_2, \ldots$ are i.i.d. normal random variables with mean 0 and variance 1. Then, by Lemma 1,

$$C_\alpha \varepsilon^{-1/\alpha} - 1 \leq E(N_{\infty}(\alpha, \varepsilon)) \leq C_\alpha \varepsilon^{-1/\alpha} \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \varepsilon^{1/\alpha} E(N_{\infty}(\alpha, \varepsilon)) = C_\alpha.$$

Step 2. Assume that $X_1, X_2, \ldots$ and $g$ satisfy the conditions in Theorem 1.

$$E(N_{\infty}(\alpha, \varepsilon)) = \sum_{n=1}^{\infty} P(|S_n| > \varepsilon n^{1/2+\alpha})$$

$$= \sum_{n=1}^{\infty} \{P(|S_n| > \varepsilon n^{1/2+\alpha}) - P_{\Phi}(|S_n| > \varepsilon n^{1/2+\alpha})\}$$

$$+ \sum_{n=1}^{\infty} P_{\Phi}(|S_n| > \varepsilon n^{1/2+\alpha}),$$

where $P_{\Phi}$ denotes the measure induced by the standard normal random variables. By Step 1, it is sufficient to show that

$$\lim_{\varepsilon \to 0^+} \varepsilon^{1/\alpha} \sum_{n=1}^{\infty} \{P(|S_n| > \varepsilon n^{1/2+\alpha}) - P_{\Phi}(|S_n| > \varepsilon n^{1/2+\alpha})\} = 0.$$

By Lemma 2,

$$\Delta(n) = \sup_{-\infty < x < \infty} |\Phi_n(x) - \Phi(x)| \leq \frac{C_1^*}{\xi g(\sqrt{n})} \sum_{j=1}^{n} E(X_j^2 g(X_j))$$

for all $n = 1, 2, \ldots$, where $\Phi_n(x)$ is the distribution function of $S_n / \sqrt{n}$. By the assumptions that $\sum_{j=1}^{n} E(X_j^2 g(X_j)) \leq O(n)$ and $g(x) \to \infty$ as $x \to \infty$, there exists a positive integer $n_0$ such that if $n > n_0$, $\Delta(n) \leq \frac{1}{2}$. It is obvious that

$$\lim_{\varepsilon \to 0^+} \varepsilon^{1/\alpha} \sum_{j=1}^{n_0} \{P(|S_j| > \varepsilon j^{1/2+\alpha}) - P_{\Phi}(|S_j| > \varepsilon j^{1/2+\alpha})\} = 0.$$

Hence, it is sufficient to show that

$$\lim_{\varepsilon \to 0^+} \varepsilon^{1/\alpha} \sum_{n=n_0}^{\infty} \{P(|S_n| > \varepsilon n^{1/2+\alpha}) - P_{\Phi}(|S_n| > \varepsilon n^{1/2+\alpha})\} = 0.$$

But...
\[ |P(|S_n| > \varepsilon n^{1/2 + \alpha}) - P_\Phi(|S_n| > \varepsilon n^{1/2 + \alpha})| \]
\[ \leq |\Phi_n(-\varepsilon n^\alpha) - \Phi(-\varepsilon n^\alpha)| + |\Phi_n(\varepsilon n^\alpha) - \Phi(\varepsilon n^\alpha)| \]

for all \( n = 1, 2, \ldots \). So it is sufficient to show that

\[ \lim_{\varepsilon \to 0^+} \varepsilon^{1/\alpha} \sum_{n > n_0} |\Phi_n(-\varepsilon n^\alpha) - \Phi(-\varepsilon n^\alpha)| = 0. \]  

(7)

\[ \lim_{\varepsilon \to 0^+} \varepsilon^{1/\alpha} \sum_{n > n_0} |\Phi_n(\varepsilon n^\alpha) - \Phi(\varepsilon n^\alpha)| = 0. \]  

(8)

Let \( A = \{n | n > n_0, \Delta(n) < n^{-2}\} \) and \( A' = \{n | n > n_0, \Delta(n) > n^{-2}\} \). Now if \( n \in A \), then \( |\Phi_n(-\varepsilon n^\alpha) - \Phi(-\varepsilon n^\alpha)| < \varepsilon n^{-2} \). Hence

\[ \lim_{\varepsilon \to 0^+} \varepsilon^{1/\alpha} \sum_{n \in A} n^{-2} = 0. \]

Next, if \( n \in A' \), then \( 1/\Delta(n) \leq n^2 \), \( \log(1/\Delta(n)) \leq 2 \log n \), and

\[ \lim_{\varepsilon \to 0^+} \varepsilon^{1/\alpha} \sum_{n \in A'} |\Phi_n(-\varepsilon n^\alpha) - \Phi(-\varepsilon n^\alpha)| \leq \lim_{\varepsilon \to 0^+} \varepsilon^{1/\alpha} \sum_{n \in A} n^{-2} = 0. \]

Since

\[ \limsup_{n \to \infty} n^{-1} \sum_{j=1}^n E(X_j g(X_j)) < \infty, \quad 0 < \alpha \leq \frac{1}{2}, \]

and \( \sum_{n=1}^\infty (\log n)/n^{2\alpha} g(\sqrt{n}) < \infty \), it is easy to see that

\[ \lim_{\varepsilon \to 0^+} \varepsilon^{1/\alpha} \sum_{n \in A'} \frac{(2 \log n) \sum_{j=1}^n E(X_j^2 g(X_j))}{ng(\sqrt{n})(1 + \varepsilon^2 n^{2\alpha})} = 0. \]

Hence,

\[ \lim_{\varepsilon \to 0^+} \varepsilon^{1/\alpha} \sum_{n \in A'} |\Phi_n(-\varepsilon n^\alpha) - \Phi(-\varepsilon n^\alpha)| = 0. \]

Therefore, we get (7). Similarly, we can prove that (8) holds and the proof of Theorem 1, now, is complete.

**Corollary 1.** Suppose that \( X_1, X_2, \ldots \) are mutually independent random variables such that \( E(X_n) = 0, E(X_n^2) = \sigma_n^2 = 1 \), and \( E(|X_n|^{2+\delta}) \leq M < \infty \) for some positive constants \( \delta \) and \( M \) and for all \( n = 1, 2, \ldots \). Then we have
\[ \lim_{\epsilon \to 0^+} \epsilon^{1/\alpha} E\{N_\infty(\alpha, \epsilon)\} = C_\alpha, \]

where \( \frac{1}{2} - \delta/4 < \alpha \leq \frac{1}{2} \) if \( 0 < \delta < 1 \), \( \frac{1}{4} < \alpha \leq \frac{1}{2} \) if \( \delta \geq 1 \).

A sharper result is

**Corollary 2.** Suppose that \( X_1, X_2, \ldots \) are mutually independent random variables such that \( E(X_n) = 0 \), \( E(X_n^2) = \sigma_n^2 = 1 \), and \( E\left( X_n^2 (\log^+ |X_n|)^{2+\delta} \right) \leq M < \infty \) for some positive constants \( \delta \) and \( M \), and for all \( n = 1, 2, \ldots \). Then we have

\[ \lim_{\epsilon \to 0} \epsilon^2 E\{N_\infty(\frac{1}{2}, \epsilon)\} = C_{1/2} = 1. \]

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**References**