

A REMARK ON THE STRONG LAW OF LARGE NUMBERS

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ABSTRACT. Let X_1, X_2, \dots be mutually independent random variables such that $E(X_n) = 0$ and $E(X_n^2) = \sigma_n^2 = 1$ for all $n = 1, 2, \dots$. For each $n = 1, 2, \dots$, let $S_n = \sum_{j=1}^n X_j$; then, by the Kolmogorov criterion for mutually independent random variables, $S_n/n^{1/2+\alpha} \rightarrow 0$ almost surely as $n \rightarrow \infty$ for any positive constant α . A deeper understanding of this theorem will be facilitated if we know the order of magnitude of $E\{N_\infty(\alpha, \epsilon)\}$ as $\epsilon \rightarrow 0^+$, where $N_\infty(\alpha, \epsilon)$ is the integer-valued random variable defined by $N_\infty(\alpha, \epsilon) = \sum_{n=1}^\infty \chi_{(\epsilon n^{1/2+\alpha}, \infty)}(|S_n|)$. The present note does the work for a wide class of random variables by using Esseen's theorem and Katz-Petrov's theorem.

Let X_1, X_2, \dots be mutually independent random variables such that $E(X_n) = 0$ and $E(X_n^2) = \sigma_n^2 = 1$ for all $n = 1, 2, \dots$. For each $n = 1, 2, \dots$, let $S_n = \sum_{j=1}^n X_j$; then, by the Kolmogorov criterion for mutually independent random variables, $S_n/n^{1/2+\alpha} \rightarrow 0$ almost surely as $n \rightarrow \infty$ for any positive constant α . For any positive constants α and ϵ , let $A_n(\alpha, \epsilon) = \chi_{(\epsilon n^{1/2+\alpha}, \infty)}(|S_n|)$ for all $n = 1, 2, \dots$ and let $N_\infty(\alpha, \epsilon) = \sum_{n=1}^\infty A_n(\alpha, \epsilon)$. Then, for a deeper understanding of the theorem above, we will study the order of magnitude of $E\{N_\infty(\alpha, \epsilon)\}$ as $\epsilon \rightarrow 0^+$, and this is just the main purpose of this note. We start with the following useful lemmas.

LEMMA 1. Suppose that X_1, X_2, \dots are mutually independent, standard normal random variables, and α, ϵ are two positive constants. Then, we have

$$(1) \quad C_\alpha \epsilon^{-1/\alpha} - 1 \leq E\{N_\infty(\alpha, \epsilon)\} \leq C_\alpha \epsilon^{-1/\alpha},$$

where $C_\alpha = \pi^{-1/2} 2^{1/2\alpha} \Gamma(1/2 + 1/2\alpha)$.

PROOF. For any positive integer m , let $N_m(\alpha, \epsilon) = \sum_{n=1}^m A_n(\alpha, \epsilon)$; then $1 + E\{N_m(\alpha, \epsilon)\} = 2 \sum_{n=0}^m \Phi(-\epsilon n^\alpha)$, where $\Phi(x)$ is the distribution function of the standard normal random variable. By the Euler-Maclaurin sum formula (see [1, pp. 124-125]),

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$$\begin{aligned} 2 \sum_{n=0}^m \Phi(-\varepsilon n^\alpha) \\ = \frac{1}{2} + \Phi(-\varepsilon m^\alpha) + 2 \int_0^m \Phi(-\varepsilon x^\alpha) dx - 2 \int_0^m R(x) d\Phi(-\varepsilon x^\alpha), \end{aligned}$$

where $R(x) = [x] - x + 1/2$ and $[x]$ denotes the greatest integer not exceeding x . Letting $m \rightarrow \infty$ and by the monotone convergence theorem, we get

$$\frac{1}{2} + E\{N_\infty(\alpha, \varepsilon)\} = 2 \int_0^\infty \Phi(-\varepsilon x^\alpha) dx - 2 \int_0^\infty R(x) d\Phi(-\varepsilon x^\alpha),$$

hence

$$\begin{aligned} 2 \int_0^\infty \Phi(-\varepsilon x^\alpha) dx + \int_0^\infty d\Phi(-\varepsilon x^\alpha) \\ \leq \frac{1}{2} + E\{N_\infty(\alpha, \varepsilon)\} \\ \leq 2 \int_0^\infty \Phi(-\varepsilon x^\alpha) dx - \int_0^\infty d\Phi(-\varepsilon x^\alpha). \end{aligned}$$

Therefore,

$$c_\alpha \varepsilon^{-1/\alpha} - 1 \leq E\{N_\infty(\alpha, \varepsilon)\} \leq C_\alpha \varepsilon^{-1/\alpha}.$$

For Lemmas 2 and 3, let X_1, X_2, \dots be mutually independent random variables such that $E(X_n) = 0$, $E(X_n^2) = \sigma_n^2 < \infty$ (not all of σ_n^2 's are zero), $S_n = \sum_{j=1}^n X_j$, $B_n^2 = \sum_{j=1}^n \sigma_j^2$, $\Phi_n(x) = P\{S_n/B_n \leq x\}$ for all $n = 1, 2, \dots$, and let G be the class of nonnegative functions $g(x)$ satisfying the following conditions: (i) $g(x)$ is nondecreasing on the interval $(0, \infty)$, is even on $(-\infty, \infty)$ and $g(x) \rightarrow \infty$ as $x \rightarrow \infty$; (ii) the function $x/g(x)$ does not decrease on $(0, \infty)$.

LEMMA 2 (PETROV). *If there exists a function g in G such that $E\{X_n^2 g(X_n)\} < \infty$ for all $n = 1, 2, \dots$, then there exists an absolute constant C_1^* such that*

$$(2) \quad \Delta(n) = \sup_{-\infty < x < \infty} |\Phi_n(x) - \Phi(x)| \leq \frac{C_1^*}{B_n^2 g(B_n)} \sum_{j=1}^n E\{X_j^2 g(X_j)\}.$$

PROOF. See [4, pp. 242–244].

LEMMA 3 (ESSEEN). *If $\Delta(n) \leq \frac{1}{2}$ for all $n > n_0$, then there exists an absolute constant C_2^* such that*

$$(3) \quad |\Phi_n(x) - \Phi(x)| \leq \min \left\{ \Delta(n), C_2^* \cdot \frac{\Delta(n) \log(1/\Delta(n))}{1 + x^2} \right\},$$

for all $n > n_0$ and all values of x .

PROOF. See [2, pp. 68–70].

Now we are in the position to state and prove our main theorem and corollaries.

THEOREM 1. Suppose that X_1, X_2, \dots are mutually independent random variables such that $E(X_n) = 0$ and $E(X_n^2) = \sigma_n^2 = 1$ for all $n = 1, 2, \dots$. If there exists a function g in G such that

- (i) $\limsup_{n \rightarrow \infty} n^{-1} \sum_{j=1}^n E\{X_j^2 g(X_j)\} < \infty$,
- (ii) $\sum_{n=1}^{\infty} (\log n)/n^{2\alpha} g(\sqrt{n}) < \infty$ for some constant α ($0 < \alpha \leq \frac{1}{2}$), then, we have

$$(4) \quad \lim_{\epsilon \rightarrow 0^+} \epsilon^{1/\alpha} E\{N_{\infty}(\alpha, \epsilon)\} = C_{\alpha}.$$

PROOF. The proof will be given by the following two steps.

Step 1. Assume that X_1, X_2, \dots are i.i.d. normal random variables with mean 0 and variance 1. Then, by Lemma 1,

$$C_{\alpha} \epsilon^{-1/\alpha} - 1 \leq E\{N_{\infty}(\alpha, \epsilon)\} \leq C_{\alpha} \epsilon^{-1/\alpha} \quad \text{and} \quad \lim_{\epsilon \rightarrow 0^+} \epsilon^{1/\alpha} E\{N_{\infty}(\alpha, \epsilon)\} = C_{\alpha}.$$

Step 2. Assume that X_1, X_2, \dots and g satisfy the conditions in Theorem 1.

$$\begin{aligned} E\{N_{\infty}(\alpha, \epsilon)\} &= \sum_{n=1}^{\infty} P\{|S_n| > \epsilon n^{1/2+\alpha}\} \\ &= \sum_{n=1}^{\infty} \{P(|S_n| > \epsilon n^{1/2+\alpha}) - P_{\Phi}(|S_n| > \epsilon n^{1/2+\alpha})\} \\ &\quad + \sum_{n=1}^{\infty} P_{\Phi}(|S_n| > \epsilon n^{1/2+\alpha}), \end{aligned}$$

where P_{Φ} denotes the measure induced by the standard normal random variables. By Step 1, it is sufficient to show that

$$(5) \quad \lim_{\epsilon \rightarrow 0^+} \epsilon^{1/\alpha} \sum_{n=1}^{\infty} \{P(|S_n| > \epsilon n^{1/2+\alpha}) - P_{\Phi}(|S_n| > \epsilon n^{1/2+\alpha})\} = 0.$$

By Lemma 2,

$$\Delta(n) = \sup_{-\infty < x < \infty} |\Phi_n(x) - \Phi(x)| \leq \frac{C_1^*}{ng(\sqrt{n})} \sum_{j=1}^n E\{X_j^2 g(X_j)\}$$

for all $n = 1, 2, \dots$ where $\Phi_n(x)$ is the distribution function of S_n/\sqrt{n} . By the assumptions that $\sum_{j=1}^n E\{X_j^2 g(X_j)\} \leq O(n)$ and $g(x) \rightarrow \infty$ as $x \rightarrow \infty$, there exists a positive integer n_0 such that if $n > n_0$, $\Delta(n) \leq \frac{1}{2}$. It is obvious that

$$\lim_{\epsilon \rightarrow 0^+} \epsilon^{1/\alpha} \sum_{j=1}^{n_0} \{P(|S_j| > \epsilon j^{1/2+\alpha}) - P_{\Phi}(|S_j| > \epsilon j^{1/2+\alpha})\} = 0.$$

Hence, it is sufficient to show that

$$(6) \quad \lim_{\epsilon \rightarrow 0^+} \epsilon^{1/\alpha} \sum_{n>n_0} \{P(|S_n| > \epsilon n^{1/2+\alpha}) - P_{\Phi}(|S_n| > \epsilon n^{1/2+\alpha})\} = 0.$$

But

$$\begin{aligned} |P(|S_n| > \varepsilon n^{1/2+\alpha}) - P_\Phi(|S_n| > \varepsilon n^{1/2+\alpha})| \\ &\leq |\Phi_n(-\varepsilon n^\alpha) - \Phi(-\varepsilon n^\alpha)| + |\Phi_n(\varepsilon n^\alpha) - \Phi(\varepsilon n^\alpha)| \end{aligned}$$

for all $n = 1, 2, \dots$. So it is sufficient to show that

$$(7) \quad \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{1/\alpha} \sum_{n > n_0} |\Phi_n(-\varepsilon n^\alpha) - \Phi(-\varepsilon n^\alpha)| = 0.$$

$$(8) \quad \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{1/\alpha} \sum_{n > n_0} |\Phi_n(\varepsilon n^\alpha) - \Phi(\varepsilon n^\alpha)| = 0.$$

Let $A = \{n | n > n_0, \Delta(n) \leq n^{-2}\}$ and $A' = \{n | n > n_0, \Delta(n) > n^{-2}\}$. Now if $n \in A$, then $|\Phi_n(-\varepsilon n^\alpha) - \Phi(-\varepsilon n^\alpha)| \leq \Delta(n) \leq n^{-2}$. Hence

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^{1/\alpha} \sum_{n \in A} |\Phi_n(-\varepsilon n^\alpha) - \Phi(-\varepsilon n^\alpha)| \leq \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{1/\alpha} \sum_{n \in A} n^{-2} = 0.$$

Next, if $n \in A'$, then $1/\Delta(n) \leq n^2$, $\log(1/\Delta(n)) \leq 2 \log n$, and

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0^+} \left\{ \varepsilon^{1/\alpha} \sum_{n \in A'} |\Phi_n(-\varepsilon n^\alpha) - \Phi(-\varepsilon n^\alpha)| \right\} \\ &\leq \lim_{\varepsilon \rightarrow 0^+} \left\{ \varepsilon^{1/\alpha} \sum_{n \in A'} \frac{C_2^* \Delta(n) 2 \log n}{1 + \varepsilon^2 n^{2\alpha}} \right\} \\ &\leq \lim_{\varepsilon \rightarrow 0^+} \left\{ \varepsilon^{1/\alpha} \sum_{n \in A'} \frac{C_1^* C_2^* 2 \log n \sum_{j=1}^n E\{X_j^2 g(X_j)\}}{ng(\sqrt{n}) \{1 + \varepsilon^2 n^{2\alpha}\}} \right\} \end{aligned}$$

(the first inequality is implied by Lemma 3 and the second one is implied by Lemma 2).

Since

$$\lim_{n \rightarrow \infty} \sup n^{-1} \sum_{j=1}^n E\{X_j^2 g(X_j)\} < \infty, \quad 0 < \alpha \leq \frac{1}{2},$$

and $\sum_{n=1}^{\infty} (\log n)/n^{2\alpha} g(\sqrt{n}) < \infty$, it is easy to see that

$$\lim_{\varepsilon \rightarrow 0^+} \left\{ \varepsilon^{1/\alpha} \sum_{n \in A'} \frac{(2 \log n) \sum_{j=1}^n E\{X_j^2 g(X_j)\}}{ng(\sqrt{n}) \{1 + \varepsilon^2 n^{2\alpha}\}} \right\} = 0.$$

Hence,

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^{1/\alpha} \sum_{n \in A'} |\Phi_n(-\varepsilon n^\alpha) - \Phi(-\varepsilon n^\alpha)| = 0.$$

Therefore, we get (7). Similarly, we can prove that (8) holds and the proof of Theorem 1, now, is complete.

COROLLARY 1. Suppose that X_1, X_2, \dots are mutually independent random variables such that $E(X_n) = 0$, $E(X_n^2) = \sigma_n^2 = 1$, and $E(|X_n|^{2+\delta}) \leq M < \infty$ for some positive constants δ and M , and for all $n = 1, 2, \dots$. Then we have

$$(9) \quad \lim_{\epsilon \rightarrow 0^+} \epsilon^{1/\alpha} E\{N_\infty(\alpha, \epsilon)\} = C_\alpha,$$

where $\frac{1}{2} - \delta/4 < \alpha \leq \frac{1}{2}$ if $0 < \delta < 1$, $\frac{1}{4} < \alpha \leq \frac{1}{2}$ if $\delta \geq 1$.

A sharper result is

COROLLARY 2. Suppose that X_1, X_2, \dots are mutually independent random variables such that $E(X_n) = 0$, $E(X_n^2) = \sigma_n^2 = 1$, and $E\{X_n^2(\log^+|X_n|)^{2+\delta}\} \leq M < \infty$ for some positive constants δ and M , and for all $n = 1, 2, \dots$. Then we have

$$(10) \quad \lim_{\epsilon \rightarrow 0} \epsilon^2 E\{N_\infty(\frac{1}{2}, \epsilon)\} = C_{1/2} = 1.$$

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