CHARACTERISTIC NUMBERS FOR UNORIENTED SINGULAR G-BORDISM

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Abstract. We develop the notion of characteristic numbers for unoriented singular G-manifolds in a G-space, G being a finite group, and prove their invariance with respect to unoriented singular G-bordism.

Thom [5] gave the notion of Stiefel Whitney numbers and Pontrjagin numbers of a manifold $M^n$ and proved its invariance with respect to bordism. Chung N. Lee and Arthur Wasserman [4] developed these notions for G-manifolds. In this note we have developed these notions for unoriented singular principal G-manifolds in a G-space, G being a finite group, and proved their invariance with regard to unoriented singular G-bordism.

1. Characteristic numbers. Let $X$ be a finite CW-complex with free action of $G$, $G$ being a finite group, and $X/G$ be again a finite CW-complex. Let $h^*$ be an equivariant cohomology theory and $h_*$ be the associated equivariant homology theory [1]. Let $h^* = H^* \circ A$ and $h_* = H_* \circ A$, where $A$ is a functor from the category of G-spaces and equivariant maps to the category of topological spaces and continuous maps, $H^*$ is the singular cohomology theory and $H_*$ is the associated singular homology theory. Let

$$\langle \cdot, \cdot \rangle : h^*(X; G) \otimes_{H^* (pt.)} h_*(X; G) \to H_*(pt.)$$

be the Kronecker pairing.

Let us assign to each compact $G$-manifold $W$, a class

$$[W, \partial W] \in h_*(W, \partial W; G)$$

such that

(a) $[W_1 \cup W_2, \partial W_1 \cup \partial W_2] = [W_1, \partial W_1] + [W_2, \partial W_2]$,
(b) $\partial_* [W, \partial W] = [\partial W]$.

Suppose $[M^n, f; G]$ is an element of unoriented bordism group $\mathcal{MR}_n(X; G)$ [3] and $x \in h^*(B(O, G)_n; G)$, $B(O, G)_n$ being the classifying space for $G$-vector bundles of dimension $n$. Then the $x$-characteristic number of the map $f: M^n \to X$ associated with an element $a^m \in h^m(X; G)$ is defined to be
\(\langle \tau_{M^*}, f^*(a^m), [M]\rangle\), where \(\tau_{M^*}: M^n \rightarrow B(O, G)_n\) is the tangent map.

In particular, let the equivariant cohomology \(h^*\) be given by \(h^*(X; G) = H^*((E_G \times X)/G; \mathbb{Z}_2)\) and \(h_\bullet\) be the associated equivariant homology, i.e. \(h_\bullet(X; G) = H_\bullet((E_G \times X)/G; \mathbb{Z}_2)\), where the action of \(G\) on \(E_G \times X\) is given by \(g(e, x) = (ge, gx)\), \(E_G\) being the total space of the universal \(G\)-bundle. Consider the map \(q: X/G \rightarrow (E_G \times X)/G\) given by \(q([x]) = [\tilde{\alpha}(x), x]\), where \(\tilde{\alpha}\) is the map given by the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\tilde{\alpha}} & E_G \\
\downarrow & & \downarrow \\
X/G & \xrightarrow{\alpha} & BG
\end{array}
\]

The map \(q\) is homotopy equivalence. Thus

\[h^*(X; G) \cong H^*(X/G; \mathbb{Z}_2)\]

and

\[h_\bullet(X; G) \cong H_\bullet(X/G; \mathbb{Z}_2)\].

Therefore \(h_\bullet(M^n; G) \cong H_\bullet(M^n/G; \mathbb{Z}_2)\) has a topological class, say \(\sigma_n\), in dimension \(n\).

**2. Invariance of characteristic numbers.** Throughout the section we will be considering equivariant cohomology \(h^*\) to be

\[h^*(X; G) = H^*((E_G \times X)/G; \mathbb{Z}_2)\]

and equivariant homology \(h_\bullet\) to be

\[h_\bullet(X; G) = H_\bullet((E_G \times X)/G; \mathbb{Z}_2)\].

**Theorem 2.1.** If \([M^n, f; G] \in \mathfrak{R}_n(X; G)\) is zero then all the \(x\)-characteristic numbers of the map \(f: M^n \rightarrow X\) associated with every \(a^m \in h^m(X; G)\) are zero.

**Proof.** Since \([M^n, f; G] \in \mathfrak{R}_n(X; G)\) is zero, \(\exists\) an \((n + 1)\)-dimensional compact principal \(G\)-manifold \(W^{n+1}\) and an equivariant map \(F: W^{n+1} \rightarrow X\) with \(\partial W^{n+1} = M^n\) and \(F/M^n = f\). Let \(\omega_{n+1} \in h_{n+1}(W^{n+1}, \partial W^{n+1}; G)\) be the topological class of \(W^{n+1}\). Then \(\partial_\bullet(\omega_{n+1}) = \sigma_n\). We have the following commutative diagram:

\[
\begin{array}{ccc}
h^*(B(O, G)_n; G) & \xrightarrow{\tau_{M^n}} & h^*(M^n; G) \\
\uparrow j^* & & \uparrow i^* \\
h^*(B(O, G)_{n+1}; G) & \xrightarrow{\tau_{W^{n+1}}} & h^*(W^{n+1}; G),
\end{array}
\]

where \(j: B(O, G)_n \rightarrow B(O, G)_{n+1}\) is the map classifying \(\mu_n \oplus 1, \mu_n \rightarrow B(O, G)_n\) being the universal \(G\)-vector bundle. Also we have

\[h^*(B(O, G)_n; G) = H^*((E_G \times B(O, G)_n)/G; \mathbb{Z}_2) = H^*(BG \times BO_n; \mathbb{Z}_2) [6] = H^*(BG; \mathbb{Z}_2) \otimes H^*(BO_n; \mathbb{Z}_2)\]
and
\[ h_*(B(O,G)_n; G) = H_*(BG; Z_2) \otimes H_*(BO_n; Z_2). \]

Thus the map \( j^* \) is a surjection. Therefore for every \( x \in h^*(B(O,G)_n; G) \), \( \exists y \in h^*(B(O,G)_{n+1}; G) \) such that \( j^*(y) = x \). Therefore
\[
\langle \tau_{M^*}(x) j^*(a^m), \sigma_n \rangle = \langle \tau_{M^*} j^*(y) f^*(a^m), \sigma_n \rangle = \langle i^* w_{n+1} j^*(y) F^*(a^m), \sigma_n \rangle \\
= \langle i^* w_{n+1} (y) F^*(a^m), i_\ast \partial_\ast (\omega_{n+1}) \rangle = 0.
\]

This completes the proof of the theorem.

Consider now the map \( \mu : \mathcal{R}_\ast(X; G) \to h_*(X; G) \) defined as \( \mu([M^n,f; G]) = q_\ast j_\ast (\sigma_n) \), where \( j \) is the map given by the following commutative diagram:
\[
\begin{array}{ccc}
M^n & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
M^n/G & \xrightarrow{j} & X/G
\end{array}
\]

\( \sigma_n \in H_n(M^n/G; Z_2) \) being the fundamental class and let \( \bar{\mu} : \mathcal{R}_\ast(X/G) \to H_\ast(X/G; Z_2) \) be the map defined by \( \bar{\mu}([N^n,g]) = g_\ast (\sigma'_n) \), where \( \sigma'_n \in H_n(N^n; Z_2) \) is the fundamental class. Suppose \( \phi_\ast : \mathcal{R}_\ast(X; G) \to \mathcal{R}_\ast(X/G) \) is the isomorphism [3] defined as \( \phi_\ast([M^n,f; G]) = [M^n/G, f] \). Then \( \mu = q_\ast \bar{\mu} \phi_\ast \) and, therefore, \( \mu \) is an epimorphism, since \( \bar{\mu} \) is so [2]. For every \( a \in h_\ast(X; G) \), we select \([M^n,f; G] \in \mathcal{R}_\ast(X; G)\) such that \( \mu([M^n,f; G]) = a \).

We define the \( \mathcal{R} \)-module structure on \( h_\ast(X; G) \) by
\[
[V^m]a = \mu[M^n \times V^m, f'; G],
\]
for every \([V^m] \in \mathcal{R} \), where the action of \( G \) on \( M^n \times V^m \) is defined as \( g(x,y) = (gx,y) \) and \( f' : M^n \times V^m \to X \) is defined as \( f'(x,y) = f(x) \). Thus \( h_\ast(X; G) \otimes \mathcal{R} \) is a \( \mathcal{R} \)-module. Let \( \{C_{n,i}\} \) be the additive base of \( h_\ast(X; G) \). Let \([M_{n,i}^n, f_i ; G] \in \mathcal{R}_\ast(X; G) \) with \( \mu([M_{n,i}^n, f_i ; G]) = C_{n,i} \). We define \( h : h_\ast(X; G) \otimes \mathcal{R} \to \mathcal{R}_\ast(X; G) \) by \( h(C_{n,i} \otimes 1) = [M_i^n, f_i ; G] \).

**Theorem 2.2.** The map \( h : h_\ast(X; G) \otimes \mathcal{R} \to \mathcal{R}_\ast(X; G) \), defined as above is an isomorphism.

**Proof.** We have the following commutative diagram:
\[
\begin{array}{ccc}
h_\ast(X; G) \otimes \mathcal{R} & \xrightarrow{h} & \mathcal{R}_\ast(X; G) \\
\downarrow q^{-1}_\ast \otimes 1_\mathcal{R} & \downarrow & \downarrow \phi_\ast \\
H_\ast(X/G; Z_2) \otimes \mathcal{R} & \xrightarrow{\overline{h}} & \mathcal{R}_\ast(X/G)
\end{array}
\]

where \( \overline{h} : H_\ast(X/G; Z_2) \otimes \mathcal{R} \to \mathcal{R}_\ast(X/G) \) is defined as \( \overline{h}(C_{n,i} \otimes 1) = [M_i^n/G, f_i] \), where \( C_{n,i} = q^{-1}_\ast(C_{n,i}) \). We already know that \( \overline{h} \) is an isomorphism [2] and, therefore, so is \( h \).

The above theorem gives the converse of Theorem 2.1 given as below.
Theorem 2.3. If all the characteristic numbers of an element \([M^n,f; G]\) \(\in N_\ast(X; G)\) are zero, then \([M^n,f; G] = 0\).

Proof. Let \(\mu([M^n,f; G]) = C_n \in h_\ast(X; G)\) and \(q^{-1}_\ast(C_n) = \bar{C}_n \in H_\ast(X/G; \mathbb{Z}_2)\). Therefore \(f_\ast(\bar{C}_n) = \bar{C}_n\). Suppose \(\{C_{n,j}\}_{j \in I}\) is an additive base of \(h_\ast(X; G)\) and \(C_{n,j} \in h^n(X; G)\) is the cohomology class dual to \(C_{n,j}\) in the sense \(\langle C_{n,j}, \bar{C}_{n,j} \rangle = \delta_{ij}\), where \(q^{-1}_\ast(C_{n,j}) = \bar{C}_{n,j}\) and \(q_\ast(C_{n,j}) = \bar{C}_{n,j}\). Let \(C_n = \sum_{j \in S} \pm C_{n,j}\), \(S\) being a finite subset of \(I\). Then if \(C_n = \sum_{j \in S} \pm C_{n,j}\), by hypothesis the \(x\)-characteristic number of \([M^n,f; G]\) associated with \(C_n \in h^n(X; G)\) is zero, that means taking \(x\) to be unit class of \(h^\ast(B(O,G)\wedge; G)\),

\[
\langle f_\ast(C_n), [M] \rangle = 0, \quad \text{or} \quad \langle f_\ast(C_n), q_\ast[\bar{C}_n] \rangle = 0,
\]

or

\[
\langle (q^{-1}_\ast(C_n))^\ast, f_\ast q_\ast[\bar{C}_n] \rangle = 0, \quad \text{or} \quad \langle (q^{-1}_\ast(C_n))^\ast, q_\ast f_\ast[\bar{C}_n] \rangle = 0,
\]

by the following commutative diagram

\[
\begin{array}{ccc}
h_\ast(M^n; G) & \xrightarrow{f_\ast} & h_\ast(X; G) \\
\downarrow q^{-1}_\ast & & \downarrow q^{-1}_\ast \\
H_\ast(M^n/G; \mathbb{Z}_2) & \xrightarrow{f_\ast} & H_\ast(X/G; \mathbb{Z}_2)
\end{array}
\]

Therefore \(\langle (q^{-1}_\ast(C_n))^\ast, q_\ast f_\ast[\bar{C}_n] \rangle = 0\), which implies that \(\langle \bar{C}_n, \bar{C}_n \rangle = 0\), showing that \(\bar{C}_n = 0\). Also it is easy to see that \(h(C_n \otimes 1) = [M^n,f; G]\). Since \(h\) is an isomorphism and \(C_n = 0\), \([M^n,f; G] = 0\), which completes the proof of the theorem.

Theorems 2.1 and 2.3 give the invariance of characteristic numbers with regard to unoriented singular principal \(G\)-bordism.

References


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