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A SINGULAR INTEGRAL INEQUALITY ON A BOUNDED INTERVAL

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ABSTRACT. An inequality of the form (1.1) is established, where p, q are real-valued functions on an interval $[a,b)$ of the real line, with $-\infty < a < b < \infty$, $p(x) > 0$ on $[a,b)$, μ_0 is a real number and f is a complex-valued function in a linear manifold so chosen that all three integrals in (1.1) are absolutely convergent. The problem is singular in that while $p^{-1} \in L(a,b)$ we require q to have a behavior at b of such a form that $q \notin L(a,b)$.

1. **Introduction.** An inequality of the form

$$\int_a^b [p|f'|^2 + q|f|^2] \geqslant \mu_0 \int_a^b |f|^2 \qquad (f \in D) \tag{1.1}$$

is established, where p, q are real-valued functions on an interval $[a,b)$ of the real line, with $-\infty < a < b < \infty$, $p(x) > 0$ on $[a,b)$, μ_0 is a real number and f is a complex-valued function in a linear manifold so chosen that all three integrals in (1.1) are absolutely convergent. The problem is singular in that while $p^{-1} \in L(a,b)$ we require q to have a behavior at b of such a form that $q \notin L(a,b)$.

We have established, in a previous paper [1], an inequality of the form (1.1) for the regular case, i.e., p^{-1} and q integrable on $[a,b]$, and also for the singular case where $b = \infty$. Some recent work by Everitt and Giertz [4] and by Kalf [5] make it feasible to study singular inequalities of the form (1.1) on bounded half-open intervals.

The Euler equation for minimizing the left-hand side of (1.1) is

$$M[y] = \lambda y \quad \text{on } [a,b), \tag{1.2}$$

where λ is a parameter and $M[y]$ is the second-order linear differential expression

$$M[y] = -(py')' + qy \quad \text{on } [a,b) \qquad ('\equiv d/dx).$$

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We make use of certain well-known relationships between equation (1.2) and inequality (1.1) from the calculus of variations, but we do not require that the functions in D for which (1.1) holds satisfy specific boundary conditions at the endpoints a and b as is common when such problems are considered by methods in the calculus of variations.

We use the following notations: $L(a,b)$ and $L^2(a,b)$ denote the classical Lebesgue complex integration spaces; a property is 'loc' on $[a,b)$ if it is satisfied on all compact subintervals of $[a,b)$; AC represents absolute continuity. Thus $\mathrm{AC}_{\mathrm{loc}}\,[a,b)$ is the class of all functions that are absolutely continuous on compact subintervals of $[a,b)$.

The coefficient functions p and q are required to satisfy the following basic conditions:

$$(1.3)\qquad \begin{aligned} &\text{(i)}\quad p \in \mathrm{AC}_{\mathrm{loc}}\,[a,b),\ p(x) > 0,\ (x \in [a,b));\\ &\text{(ii)}\quad \text{both } p' \text{ and } q \text{ belong to } L^2_{\mathrm{loc}}[a,b). \end{aligned}$$

Note that these conditions imply that the differential expression M is regular at all points of $[a,b)$. (See [6, §16.1].) We remark below on the need for (ii).

Following the notation in [1, §2] we define the following sets of functions:

$$(1.4)\qquad \begin{aligned} &\text{(i)}\quad \Delta = \{f \in L^2(a,b)\colon f' \in \mathrm{AC}_{\mathrm{loc}}\,[a,b), M[f] \in L^2(a,b)\};\\ &\text{(ii)}\quad \text{for } \alpha \in [0,\pi),\ \mathfrak{D}(\alpha) = \{f \in \Delta\colon f(a)\cos\alpha + f'(a)\sin\alpha = 0\};\\ &\text{(iii)}\quad D = \{f \in L^2(a,b)\colon f \in \mathrm{AC}_{\mathrm{loc}}\,[a,b), p^{1/2}f', |q|^{1/2}f \in L^2(a,b)\}. \end{aligned}$$

It should be noted that (ii)(1.3) implies that Δ contains all infinitely differentiable functions with compact support in (a,b) and so Δ is dense in $L^2(a,b)$.

For each $\alpha \in [0,\pi)$, an operator $T(\alpha)$ is defined by

$$\textit{domain of } T(\alpha) \textit{ is } \mathfrak{D}(\alpha) \textit{ and } T(\alpha)f = M[f].$$

It is known that $T(\alpha)$ is selfadjoint in $L^2(a,b)$ if, and only if, M is limit-point at the singular endpoint b. (See [6, §18.3].)

Additionally we assume the following conditions on the coefficient functions p and q:

$$(1.5)\qquad \begin{aligned} &\text{(i)}\quad p^{-1} \in L(a,b);\\ &\text{(ii)}\quad \int_a^b q_+ = \infty, \text{ where } q_+ = (q + |q|)/2. \end{aligned}$$

Both conditions are needed in the proof of our theorem. The second condition (ii) insures that $q \notin L(a,b)$ and so forces b to be a singular endpoint for the differential expression M [6, §16.1]. This then is a distinct departure from the work contained in [1, §2].

Finally, the following conditions are required:

(1.6)

(i) M satisfies the Dirichlet condition at b, i.e.,

$p^{1/2}f'$ and $|q|^{1/2}f \in L^2(a,b)$ for all $f \in \Delta$;

(ii) the operator $T(\frac{1}{2}\pi)$ is bounded below in $L^2(a,b)$;

i.e., there is a real number μ_0 such that

$(T(\frac{1}{2}\mu)f,f) \geqslant \mu_0(f,f)$ for all $f \in \mathfrak{D}(\frac{1}{2}\pi)$,

where $(\cdot,\cdot)$ is the usual inner product in $L^2(a,b)$.

To be exact we define $\mu_0 = \inf\{\lambda\colon \lambda$ is in the spectrum of $T(\frac{1}{2}\pi)\}$, so that condition (ii) of (1.6) is equivalent to the assumption that $\mu_0 > -\infty$.

Specific conditions on the coefficients p and q to insure that M and $T(\frac{1}{2}\pi)$ satisfy (i) and (ii) of (1.6) may be found in the papers of Everitt and Giertz [4] and Kalf [5]. The results of these papers make it reasonable to assume (1.6) as a set of conditions to be satisfied and so indirectly impose conditions on the coefficients p and q.

In [4] it is assumed that $p = 1$ on $[a,b)$ and that q satisfies a growth condition near b which insures (ii) of (1.5), (i) and (ii) of (1.6) are satisfied. In [5] a general condition is given that insures that (1.6) is satisfied, but it is then necessary to require q to satisfy (ii) of (1.5).

In [5] Kalf has shown that conditions (ii) of (1.5) and (i) of (1.6) imply that M is strong limit-point at the singular endpoint b, i.e.,

$$\lim_{x\to b^-} p(x)f(x)g'(x) = 0 \qquad (f,g \in \Delta). \tag{1.7}$$

An alternative proof of this result may be found in the paper by Everitt [3]. Note that (1.7) implies that M is limit-point at b and so all the operators $T(\alpha)$ $(\alpha \in [0,\pi))$ are selfadjoint in $L^2(a,b)$.

We can now state the main result of this paper, which is,

THEOREM 1. *If p and q are real-valued functions for which conditions* (1.5) *and* (1.6) *hold, then inequality* (1.1) *is valid for all functions f in the set D described in* (1.4)(iii), *with μ_0 the smallest number in the spectrum of the operator $T(\frac{1}{2}\pi)$.*

If μ_0 is in the point or point-continuous spectrum of $T(\frac{1}{2}\pi)$, then there is equality in (1.1) *if, and only if, $f = c\psi_0$ where c is a complex number and ψ_0 is an eigenfunction for $T(\frac{1}{2}\pi)$ corresponding to μ_0.*

If μ_0 is in the continuous spectrum of $T(\frac{1}{2}\pi)$, then there is equality in (1.1) *if, and only if, f is the zero function. The inequality is the best possible in the sense that there is a sequence $\{f_n\}$ such that $f_n \in D$, $\int_a^b |f_n|^2 = 1$ $(n = 1,2,\dots)$ and*

$$\lim_{n\to\infty} \int_a^b [p|f_n'|^2 + q|f_n|^2] = \mu_0.$$

Our proof of Theorem 1 depends upon the following approximation theorem.

THEOREM 2. *If the hypothesis of Theorem* 1 *holds, then for each $\varepsilon > 0$ and each real-valued function f in D there is a real-valued function g in $\mathfrak{D}(\frac{1}{2}\pi)$ such that*

$$\left|\int_a^b pf'^2 - \int_a^b pg'^2\right| < \varepsilon, \quad \left|\int_a^b f^2 - \int_a^b g^2\right| < \varepsilon, \quad \left|\int_a^b qf^2 - \int_a^b qg^2\right| < \varepsilon.$$

The proofs of these theorems are in the next section.

2. **Proofs.** We restrict our attention in this section to real-valued functions since it is sufficient to prove (1.1) for all real-valued functions in the set D; this is a consequence of taking the coefficients p and q to be real-valued on $[a, b)$.

We use the following lemma in our proof of Theorem 2; see also [1, §3].

LEMMA. *If the hypothesis of Theorem* 1 *holds*, $f \in D$ *and* $\varepsilon_1 > 0$, *then there is a number* X *in* (a, b) *with the property that*

$$\int_X^b pf'^2 < \varepsilon_1, \quad \int_X^b |q| f^2 < \varepsilon_1, \quad \int_X^b f^2 < \varepsilon_1, \tag{2.1}$$

and

$$f^2(X)\int_a^X |q| < \varepsilon_1, \quad (X - a) f^2(X) < \varepsilon_1. \tag{2.2}$$

PROOF. The fact that (2.1) holds follows from the fact that $f \in D$. To obtain (2.2) we first observe that f in D and (1.6)(i) imply that $\lim_{x \to b^-} f(x)$ exists. Indeed,

$$f(x) = f(a) + \int_a^x f' = f(a) + \int_a^x p^{-1/2} p^{1/2} f';$$

the last integral converges as $x \to b^-$ since $p^{-1/2}$ and $p^{1/2} f' \in L^2(a, b)$.

If $\lim_{x \to b^-} f(x) \neq 0$, then there exist numbers $k > 0$ and $t \in [a, b)$ such that $f^2(x) \geqslant k$ for $x \in (t, b)$. Then

$$\int_a^x q^+ f^2 \geqslant \int_a^t q^+ f^2 + k \int_t^x q^+.$$

This inequality and (1.5)(ii) imply that $\int_a^b q^+ f^2 = \infty$ which is contrary to the assumption that $|q|^{1/2} f \in L^2$; i.e., that $f \in D$. Therefore $\lim_{x \to b^-} f(x) = 0$.

It now follows that $(x - a) f^2(x) \to 0$ as $x \to b$. To complete the proof of the lemma it is sufficient to show that there is a sequence $\{x_n\}$, $x_n \to b$, for which $f^2(x_n) \int_a^{x_n} |q| \to 0$ as $n \to \infty$. If no such sequence exists, then there is a $d > 0$ and $t \in (a, b)$ such that

$$f^2(x) \geqslant d \Big/ \int_a^x |q| \qquad (x \in (t, b)).$$

Multiplying by $|q|$ and integrating from t, we obtain

$$\int_t^x |q| f^2 \geqslant d\left[\log \int_a^x |q| - \log \int_a^t |q|\right].$$

This inequality, (1.5)(ii) and (1.6)(i) are incompatible with $f \in D$.

PROOF OF THEOREM 2. For a positive number ε_1 we choose X in (a, b) so that the conclusion of the lemma is valid. Then we note that $f \in D$ implies that

$f' \in L^2(a,X)$ and that the set of continuously differentiable functions vanishing together with their derivatives at a and X is dense in $L^2(a,X)$. Therefore, for $\eta > 0$, we may choose a continuously differentiable function ϕ such that $\phi(a) = \phi'(a) = \phi(X) = \phi'(X) = 0$, $\phi(x) = 0$ on $[X,b)$ and $\int_a^X |f' - \phi|^2 < \eta$. The function g is defined by

$$g(x) = -\int_x^X \phi \qquad (x \in [a,X]),$$

$$g(x) = 0 \qquad (x \in (X,b)).$$

It is clear that $g \in \mathfrak{D}(\frac{1}{2}\pi)$ since $g'(a) = 0$, g has a continuous second derivative, and p, p' and q are all in $L^2(a,X)$, in view of the conditions (1.3). We note here that conditions (1.3) are essential to our argument, as were similar conditions in [1, §3], and that it remains undecided whether condition (ii) of (1.3) could be replaced by the weaker assumption that p' and q are locally integrable.

Using the function g defined above and the lemma of this section, the estimates obtained in [1, §3] remain valid, with ∞ replaced by b, and the proof of Theorem 2 proceeds in the same way as the proof of Theorem 3 of [1].

PROOF OF THEOREM 1. The proof of Theorem 1 is accomplished in three stages. The first is to establish inequality (1.1) for functions f in $\mathfrak{D}(\frac{1}{2}\pi)$, the second is to extend the validity of the inequality to all of D, and the third is to determine the cases of equality. Each of these steps is similar to a corresponding part of the proof of Theorem 2 of [1] and we therefore limit our discussion here to a statement of the basic ideas involved and points where something needs to be added to the previous argument.

We begin by showing that (1.1) holds for f in $\mathfrak{D}(\frac{1}{2}\pi)$. Indeed, for such an f an integration by parts, an application of (1.7) and condition (1.6) yield

$$\int_a^b [pf'^2 + qf^2] = \int_a^b fM[f] \geqslant \mu_0 \int_a^b f^2,$$

which establishes (1.1) for f in $\mathfrak{D}(\frac{1}{2}\pi)$. Since this argument appears to be identical to the one used in [1] it should be pointed out that an analogue of (1.7) is used in [1] and the arguments used in proving this analogue cannot be used in proving (1.7).

To prove that (1.1) holds for functions in D, we assume there is a function in D for which (1.1) fails to hold and use the results of Theorem 2 to obtain a contradiction of the fact that (1.1) holds for all functions in $\mathfrak{D}(\frac{1}{2}\pi)$. The details are the same as those used in [1].

The third stage of the proof, that of determining the cases of equality, separates into two cases according as μ_0 is an eigenvalue or not.

If μ_0 is in the point spectrum or point-continuous spectrum, then it is clear that there is equality in (1.1) for any constant multiple of an eigenfunction corresponding to μ_0. Conversely, if equality holds in (1.1) for some f in D, then it follows from Theorem 4 of [1] that f is a solution of the differential equation $M[f] = \mu_0 f$, and since the differential expression M is in the limit point case at b it follows that $f = c\psi_0$, where ψ_0 is an eigenfunction corresponding to μ_0 and c is a constant.

If μ_0 is in the continuous spectrum, we assume there is a function f in D, $f \not\equiv 0$ for which equality holds. Again using Theorem 4 of [1] we find that f is a solution of equation (1.2) for $\lambda = \mu_0$ and apply (1.7), Dirichlet condition (1.6)(i) and an integration by parts to conclude that $f(a)f'(a) = 0$. Since μ_0 is not an eigenvalue for $T(\frac{1}{2}\pi)$, $f'(a) \neq 0$; therefore, $f(a) = 0$ and μ_0 is an eigenvalue of the operator $T(0)$. That this is impossible is established as in [1] using analytic properties of the Weyl m-coefficients $m(\lambda, 0)$ and $m(\lambda, \frac{1}{2}\pi)$ presented by Chaudhuri and Everitt in [2] since those properties hold equally well for a differential expression $M[f]$ considered on a bounded interval $[a, b)$ with a singular endpoint b.

3. **Examples.** We conclude with an example. Consider inequality (1.1) with

$$a = 0, \quad b = 1, \quad p(x) = 1, \quad q(x) = 3/4(1-x)^2 \qquad (x \in [0, 1)). \tag{3.1}$$

Then the conditions imposed by Everitt and Giertz [4] are satisfied as is the Dirichlet condition (1.6)(i) at the singular point 1. Therefore, the differential expression $M[f]$ for this example is strong limit-point at 1. Also, $\int_0^1 q_+ = \int_0^1 q = \infty$. Thus all the conditions are satisfied and (1.1) holds for the example described by (3.1).

We write the differential equation (1.2) as

$$-y'' + (\nu^2 - 1/4)(1-x)^{-2} y = \lambda y \qquad (0 \leqslant x < 1), \tag{3.2}$$

where $\nu = 1$. Then the Weyl m-coefficient $m(\lambda, \frac{1}{2}\pi)$ is given by

$$m(\lambda, \tfrac{1}{2}\pi) = 2J_1(s)/(J_1(s) + 2sJ_1'(s)),$$

where $s^2 = \lambda$, $0 \leqslant \arg \lambda < 2\pi$, $0 \leqslant \arg s < \pi$, and J_1 is the Bessel function of order 1 of the first kind. (See Titchmarsh [7, §4.8].) We note that m is meromorphic and therefore the spectrum is discrete. Moreover, $\mu_0^{1/2}$ is the first positive zero of $J_1(s) + 2sJ_1'(s)$. The equalizing function (eigenfunction) is then given by

$$\psi(\gamma) = (1-x)^{1/2} J_1(s(1-x)) \qquad (0 \leqslant x < 1).$$

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