CENTRAL SEQUENCES IN FLOWS ON
2-MANIFOLDS OF FINITE GENUS

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Abstract. Let \( \phi \) be a continuous flow on the metric space \( X \) and let \( X^1, X^2, \ldots \) denote the “central” sequence of closed \( \phi \)-invariant subsets of \( X \) obtained by iterating the process of taking nonwandering points of \( \phi \). A. Schwartz and E. Thomas have proved that, if \( X \) is an orientable 2-manifold of finite genus, then this sequence can have not more than two distinct elements. We extend this result to include the nonorientable case; then this sequence can have at most three distinct elements. Analogous results are derived for the sequences obtained by iterating the processes of taking \( \alpha \) and \( \omega \) limit sets, or closures of \( \alpha \) and \( \omega \) limit sets.

1. Introduction. In [6] Schwartz and Thomas determine bounds on the lengths of the various “central sequences” in an arbitrary continuous flow on an orientable 2-manifold of finite genus. The purpose of this note is to extend these results to include the nonorientable case.

Before stating these results precisely we need several definitions. Let \( X \) be a metric space, and let \( \phi: X \times \mathbb{R} \to X \) be a continuous flow on \( X \). For \( t \in \mathbb{R} \) let \( \phi_t \) denote the homeomorphism of \( X \) defined by \( \phi_t(x) = \phi(x, t) \) (\( x \in X \)). A point \( x \in X \) is nonwandering if \( x \in \bigcup J^+(x) \); here \( J^+(x) \) denotes the set of all limits of sequences \( \{\phi(x_n, t_n)\} \), where \( \{x_n\} \) converges to \( x \) and \( \{t_n\} \) tends to infinity. If \( Y \subseteq X \) is \( \phi \)-invariant (i.e., \( \phi_t(Y) = Y \) for all \( t \in \mathbb{R} \)) then \( y \in Y \) is nonwandering relative to \( Y \) if \( y \) is a nonwandering point of the flow \( (Y, \phi|_Y) \); we let \( Y' \) denote the set of points that are nonwandering relative to \( Y \).

Definition 1 (Birkhoff [1]). Define recursively: \( X^0 = X \); \( X^{\alpha + 1} = (X^\alpha)' \) if \( \alpha \) is an ordinal and \( X^\alpha \) is defined; and \( X^\alpha = \bigcap_{\beta < \alpha} X^\beta \) if \( \alpha \) is a limit ordinal and \( X^\beta \) is defined for all \( \beta < \alpha \). The transfinite sequence \( \alpha \to X^\alpha \) of nested, closed, \( \phi \)-invariant subsets of \( X \) is called the central sequence of \( \phi \); the least ordinal \( \alpha \) such that \( X^{\alpha + 1} = X^\alpha \) is called the depth of the central sequence, and is denoted \( d(\phi) \).

A point \( y \in X \) is called an \( \omega \)-limit point (\( \alpha \)-limit point) of \( x \in X \) if there is a sequence of real numbers \( t_k \to \infty \) (\( t_k \to -\infty \)) such that \( \phi_{t_k}(x) \to y \). The set of \( \omega \)-limit points (\( \alpha \)-limit points) of \( x \) is denoted \( \Omega(x) \) (\( A(x) \)). A point \( x \in X \) is said to be \( P^+ \) stable (\( P^- \) stable) if \( x \in \Omega(x) \) (\( x \in A(x) \)); \( x \) is Poisson stable if \( x \in \Omega(x) \cap A(x) \). If \( Y \subseteq X \) is \( \phi \)-invariant let \( Y^* \) denote the set of points of \( Y \) which are either \( \alpha \)- or \( \omega \)-limit points of some point of \( Y \).

Definition 2 (Maier [3]). Define recursively:

\[
X^{[0]} = X; \quad X^{[\alpha + 1]} = \text{cl}(X^{[\alpha]} \setminus \cdot)
\]

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(here cl denotes closure) if \( \alpha \) is an ordinal and \( X^{[\alpha]} \) is defined; \( X^{[\alpha]} = \cap_{\beta < \alpha} X^{[\beta]} \) if \( \alpha \) is a limit ordinal and \( X^{[\beta]} \) is defined for all \( \beta < \alpha \). The sequence \( \alpha \to X^{[\alpha]} \) is called the 2-central sequence of \( \phi \); its depth, denoted \( d_2(\phi) \), is the least ordinal \( \alpha \) such that \( X^{[\alpha + 1]} = X^{[\alpha]} \).

**Definition 3 (Birkhoff-Smith [2]).** Define recursively: \( X^{(0)} = X; X^{(\alpha + 1)} = (X^{(\alpha)})^* \) if \( \alpha \) is an ordinal and \( X^{(\alpha)} \) is defined; \( X^{(\alpha)} = \cap_{\beta < \alpha} X^{(\beta)} \) if \( \alpha \) is a limit ordinal and \( X^{(\beta)} \) is defined for all \( \beta < \alpha \). The sequence \( \alpha \to X^{(\alpha)} \) is called the 3-central sequence of \( \phi \); its depth, denoted \( d_3(\phi) \), is the least ordinal \( \alpha \) such that \( X^{(\alpha + 1)} = X^{(\alpha)} \).

We are interested in the case in which \( X \) is a 2-manifold (separable metric, connected, and possibly with nonempty boundary) of finite genus. A 2-manifold \( X \) is said to have finite genus if there is a compact 2-manifold \( Y \subseteq X \) such that \( X - Y \) is homeomorphic to a subset of \( \mathbb{R}^2 \); in this case the genus of \( X \) is defined to be the genus of such a submanifold \( Y \) (cf. [5]). A 2-manifold is closed if it is compact and without boundary.

With this terminology the results of [6] can be stated as follows:

**Theorem (Schwartz-Thomas).** If \( X \) is an orientable 2-manifold of finite genus and \( \phi \) is a continuous flow on \( X \), then \( d_1(\phi), d_2(\phi), d_3(\phi) \leq 2 \).

If nonorientable manifolds are admitted, the corresponding result is:

**Theorem.** If \( X \) is a 2-manifold of finite genus and \( \phi \) is a continuous flow on \( X \), then \( d_1(\phi) \leq 3 \) and \( d_2(\phi), d_3(\phi) \leq 2 \).

An example is given in [6] in which \( d_1 = 3 \) and \( d_2 = d_3 = 2 \), so these bounds cannot be improved. The restriction to finite genus is necessary: in [4] it is proved that if \( \alpha, \beta, \gamma \geq 2 \) are any countable ordinals with \( \alpha \geq \beta \), then there is an orientable open 2-manifold \( X \) and a \( C^\infty \) flow \( \phi \) on \( X \), with \( d_1(\phi) = \alpha, d_2(\phi) = \beta, \) and \( d_3(\phi) = \gamma \). Also, if \( X \) is any manifold of dimension greater than two and \( \alpha, \beta, \gamma \) are as above, then there is a flow \( \phi \) on \( X \) with \( d_1(\phi) = \alpha, d_2(\phi) = \beta, \) and \( d_3(\phi) = \gamma \) [4].

2. Preliminaries. We first indicate that the “topological preliminaries” of [6] hold as well in the nonorientable case. The following proposition is an easy consequence of Theorem 3 of [5].

**Proposition 1.** Suppose that \( X \) is a 2-manifold without boundary and that \( \text{genus}(X) = n \). Then \( X \) is homeomorphic to the surface obtained by removing a closed, totally disconnected set from a closed 2-manifold of genus \( n \).

We note that a closed 2-manifold \( X \) of genus \( n \) (orientable or not) has the property that any collection of \( n + 1 \) disjoint simple closed curves in \( X \) separates \( X \). Using this fact and Proposition 1, we may prove, just as in [6]:

**Proposition 2 (Schwartz-Thomas).** Suppose that \( X \) is a closed 2-manifold. Let \( \Gamma \subseteq X \) be a continuum (i.e., compact, connected subset) which meets a closed arc \( T \) in a totally disconnected set. Let \( W \) be a component of \( X - \Gamma \) and let \( \mathcal{T} \) be the collection of components of \( T - \Gamma \) lying in \( W \). If \( \mathcal{T} \) is infinite, say \( \mathcal{T} = \{ T_i \mid i \in \mathbb{Z} \} \), then, for almost all \( i \), there is an open cell \( W_i \subseteq W \) such that the boundary of \( W_i \) is contained in \( T_i \cup \Gamma \), and \( T_i \subseteq \text{cl}(W - W_i) \).

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We list two further facts which will also be used in the proof of the main theorem. In each case the proof is straightforward and hence omitted.

Suppose that \( X \) is a 2-manifold and let \( \phi \) be a continuous flow on \( X \). Let \( T \) be a closed arc in \( X \) such that, for some \( \epsilon > 0 \), \( \phi \) is a homeomorphism of \( T \times (-\epsilon, \epsilon) \) onto its image in \( X \). Such an arc is called a transversal to \( \phi \). We define the first return map \( f \) on \( T \) as follows: \( x \in \text{domain}(f) \) if \( \phi_t(x) \in T \) for some \( t > 0 \); in this case let \( t_0 = \min \{ t > 0 | \phi_t(x) \in T \} \) and set \( f(x) = \phi_{t_0}(x) \).

**Proposition 3.** Suppose \( \phi \) is a continuous flow on the 2-manifold \( X \) and that \( T \) is a transversal to \( \phi \) with first return map \( f \). Suppose that \( f \) is monotone increasing on \( \text{domain}(f) \) and that \( y \in \text{domain}(f) \). Then \( y \in X^1 \) only if the orbit of \( y \) is periodic.

Now suppose \( y_0 \) and \( y_1 \) are points of \( \text{domain}(f) \), and set \( y'_0 = f(y_0) \), \( y'_1 = f(y_1) \). For \( i = 0, 1 \), let \( A_i \) denote the subarc of the orbit \( O(y_i) \) through \( y_i \) from \( y_i \) to \( y'_i \). Let \( B \) (\( B' \)) denote the subarc of \( T \) from \( y_0 \) to \( y_1 \) (\( y'_0 \) to \( y'_1 \)). We say that \( A_0 \) and \( A_1 \) are parallel if there is an open disc \( W \subseteq X \) such that the boundary of \( W \) is \( A_0 \cup A_1 \cup B \cup B' \).

**Proposition 4.** Suppose that \( \phi \) is a continuous flow on the closed 2-manifold \( X \) and that \( T \) is a transversal to \( \phi \) with first return map \( f \). Let \( \{y_i\} \) be a sequence of points of \( \text{domain}(f) \); for \( i \in \mathbb{Z}^+ \) let \( y'_i = f(y_i) \) and let \( A_i \) denote the subarc of \( O(y_i) \) from \( y_i \) to \( y'_i \). Suppose that both \( \{y_i\} \) and \( \{y'_i\} \) converge monotonically. Then, for almost all \( i \), \( A_i \) and \( A_{i+1} \) are parallel.


**Proposition 5.** Suppose \( X \) is a 2-manifold of finite genus and \( \phi \) is a continuous flow on \( X \). If \( x \in X^1 \), but \( x \) is not \( P^+ \) or \( P^- \) stable, then \( A(x) \) and \( \Omega(x) \) contain no regular points.

**Proof.** We follow the proof of 6.2 in [6] until the last step. We may assume that \( X \) is closed, that \( x \in X^1 \) is not \( P^+ \) or \( P^- \) stable, but that there is a regular point \( y \) in \( \Omega(x) \). Let \( \Gamma \) denote the continuum \( \Omega(x) \), and let \( T \) be a transversal to \( \phi \), with \( y \) an interior point of \( T \). By assumption \( x \notin \Gamma \). If \( \Gamma \cap T \) is infinite, then the component \( W \) of \( X - \Gamma \) that contains \( O(x) \) meets \( T \) in an infinite collection of open arcs. By Proposition 2, \( O(x) \) crosses one of these into an open, positively (or negatively) invariant 2-cell, and this contradicts \( x \in X^1 \). Thus we may assume that in fact \( \Gamma \cap T = \{x\} \).

Hence we may choose points \( x_i \in O(x) \cap T \) so that both \( \{x_i\} \) and \( \{x'_i = f(x_i)\} \) converge monotonically to \( y \) (\( f \) denotes the first return map on \( T \)). For \( i \in \mathbb{Z}^+ \), let \( A_i \) denote the subarc of \( O(x) \) from \( x_i \) to \( x'_i \). By Proposition 4 we may assume that all the \( A_i \) are parallel.

In the case that both \( \{x_i\} \) and \( \{x'_i\} \) decrease (say) monotonically to \( y \), we see that the first return map on the subinterval \( [y, x_i] \) of \( T \) is monotone increasing. As \( x_i \in X^1 \), this leads to a contradiction of Proposition 3.

In case \( \{x_i\} \) decreases to \( y \) and \( \{x'_i\} \) increases to \( y \) we may argue as follows. Pick a sequence of points \( z_i \in O(x) \cap T \) so that \( \{z_i\} \) increases to \( y \) and \( \{z'_i = f(z_i)\} \) decreases to \( y \). For \( i \in \mathbb{Z}^+ \) let \( B_i \) denote the subarc of \( O(x) \) from \( z_i \) to \( z'_i \). We may assume all the \( B_i \) are parallel. It follows that, for some \( i \), the
first return map on the subinterval \([y, x]\) of \(T\) is monotone increasing, and we have a contradiction as before.

**Proof of the Main Theorem.** The conclusions \(d_2(\phi) < 2\) and \(d_3(\phi) < 2\) follow easily from Proposition 5. We prove \(d_4(\phi) < 3\).

If this fails for any 2-manifold of finite genus, then we can construct, as in [6], a flow on a closed 2-manifold for which it fails. Hence we assume that \(X\) is closed and that there is a point \(x \in X^3 - X^4\). Let \(T\) be a transversal to \(\phi\), with \(x\) an interior point of \(T\), and let \(f\) denote the first return map on \(T\). Since \(x \not\in X^4\) we may assume that no \(P^+\) or \(P^-\) stable orbit meets \(T\). By Proposition 5, we may assume that if \(y \in X^1 \cap T\) then \(O(y) \cap T = \{x\}\) is finite.

Pick a sequence \(\{y_i\} \subseteq T\) of points on distinct orbits of \(X^2 - X^3\), so that \(\{y_i\}\) converges monotonically to \(x\) and so that the set \(\bigcup O(y_i) \cap T\) has \(x\) as an accumulation point. Let \(\Gamma\) denote the continuum \(\limsup_{i \to \infty} O(y_i)\). Exactly as in [6] we can show that the \(y_i\) may be chosen so that no \(O(y_i)\) lies in \(\Gamma\), and hence so that \(\Gamma \cap T = \{x\}\). Thus, by choosing \(T\) sufficiently small, we may assume \(\Gamma \cap T = \{x\}\). In this case the only accumulation point of the sequence \(\{y_i = f(y_i)\}\) is \(x\), so we may assume that \(\{y_i\}\) and \(\{y_i'\}\) both converge monotonically to \(x\).

If both sequences decrease to \(x\) we obtain a contradiction just as in the proof of Proposition 5. So we assume that \(\{y_i\}\) decreases to \(x\) and \(\{y_i'\}\) increases to \(x\). Let \(\{T_i\}\) be a sequence of disjoint closed subarcs of \(T\), with \(T_i \subseteq \text{domain}(f)\) and \(y_i\) an interior point of \(T_i\). Let \(T_i' = f(T_i)\). Since \(y_i \in X^2\) there must be points \(z_i' \in T_i' \cap X^1\) such that \(z_i = f(z_i') \in T_i\). By Proposition 4 we may assume that all of the orbit segments \(A_i = [y_i, y_i']\) are parallel, and that all the orbit segments \(B_i = [z_i', z_i]\) are parallel. It follows that, for some \(n\), the first return map on the subarc \([y_i', x]\) of \(T\) is monotone increasing. Note that, for \(i\) sufficiently large, \(z_i'\) is in the domain of this map (though \(y_i'\) need not be). As \(z_i' \in X^1\), this contradicts Proposition 3.

**Figure 1**

![Diagram showing the relationship between the points and sets described in the text.](image)

**Bibliography**


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