S-LIMITS AND $\alpha$-SUMMABILITY

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Abstract. Certain classes of subspaces of $m$ are examined that are the domains of uniqueness of certain Banach type limits. These subspaces are identified as the bounded convergence domains of $\alpha$-summability. As a corollary it is shown that the closure of $bs$ in $m$ is $f_0$. Also we characterize those matrices whose convergence domains include these spaces in a manner analogous to Lorentz' characterization of strongly regular matrices.

S-limits. Throughout this paper, $S$ will denote a fixed regular matrix with $\|S\| = 1$. We shall limit our discussion to sequences and matrices of real numbers. A $S$-limit is a linear functional, $L$, on $m$, the space of bounded sequences, that satisfies: (i) $L(x) > 0$ if $x_n > 0$, $n = 1, 2, \ldots$; (ii) $L(\overline{1}) = \overline{1}$ ($\overline{1}$ is the constant sequence of 1's); (iii) $L(Sx) = L(x)$ for all $x \in m$. A Banach limit is a $T$-limit where $T$ is the translation matrix, i.e. $(Tx)_n = x_{n+1}$. A matrix $B$ is $S$-invariant if $B(S - I)$ maps every bounded sequence to a null sequence. ($I$ is the identity matrix.)

S-limits need not exist for every regular matrix $S$. Let $S = (a_{nk})$ where $a_{nk} = 0$ if $k > 2n$ and $a_{nk} = 1$ if $k = 2n$. For $1 \leq k \leq 2n$, $a_{nk} = -1/n$ if $k$ is odd, $a_{nk} = 1/n$ if $k$ is even. $S$ is regular, but for $x = \{1, 0, 1, 0, \ldots\}$, we have $Sx = \{-1, -1, -1, \ldots\}$. Now if $L$ were an $S$-limit, we would have $0 < L(x) = L(Sx) = -1$. Thus $S$-limits need not exist.

Theorem 1. If there is a nonnegative regular $S$-invariant matrix, then $S$-limits exist.

Proof. Suppose $B$ is such a matrix. Define a functional $q$ on $m$ by $q(x) = \limsup Bx$. Then $q$ is sublinear and nonnegatively homogeneous. For $x \in c$ we have, by the regularity of $B$, that $\lim x = q(x)$. By the Hahn-Banach Theorem, we can extend $\lim$ to a linear functional, $L$, on $m$ s.t. $L(x) \leq q(x)$ for all $x \in m$. Thus $-q(-x) \leq -L(-x) = L(x) \leq q(x)$, for all $x \in m$.

Since $B$ is nonnegative, for $x_n \geq 0$, $n = 1, 2, \ldots$, we have $0 \leq \liminf Bx = -q(-x) \leq L(x)$. So $L$ is nonnegative. Also for $x \in m$, $\lim B(S - I)x = 0$; thus $-q(-(S - I)x) = 0 = q((S - I)x)$ which proves $L((S - I)x) = 0$. So $L$ is an $S$-limit.

We note that the Cesàro matrix is a nonnegative regular $T$-invariant matrix;

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hence, as a corollary, we have that Banach limits exist.

For a particular $S$, the determination of the existence of an $S$-invariant matrix is facilitated by Schur's Theorem (see, for example, [8, p. 20]) which gives necessary and sufficient conditions for $B(S - I)$ to map $m$ to $n$, the space of null sequences.

In the case where $S$-limits exist for a particular regular matrix $S$, we shall consider the following sets which are the domains of uniqueness of these limits:

$$
\Lambda(S) = \{ x \in m : L(x) = L'(x) \text{ for every pair of } S\text{-limits } L, L' \},
$$

$$
\Theta(S) = \{ x \in m : L(x) = 0 \text{ for every } S\text{-limit } L \}.
$$

Note that $\Lambda(S)$ and $\Theta(S)$ are closed subspaces of $m$ since each $S$-limit $L$ is nonnegative, hence continuous.

We shall identify $\Theta(S)$ and $\Lambda(S)$, in certain cases, as the bounded convergence domains of the method of summability introduced in [3].

@-summability. For $v = 1, 2, \ldots,$ let $A^v = (a_{nk}^v)$ be an infinite matrix of real numbers. Let $\mathfrak{a}$ denote the sequence of matrices $\{A^v\}$. A sequence $x = \{x_n\}$ is $\mathfrak{a}$ or $\{A^v\}$-summable to a number $l$ if $\lim A^v x = l$ uniformly in $v = 1, 2, \ldots$. We say $\mathfrak{a}x$ converges to $l$ and write $\lim\mathfrak{a}x = l$. $\mathfrak{a}$ is said to be regular if $\mathfrak{a}x$ converges to $l$ whenever $x$ is convergent to $l$.

$$
c_{\mathfrak{a}} = \{ x : \mathfrak{a}x \text{ converges} \}
$$

$$
n_{\mathfrak{a}} = \{ x : \mathfrak{a}x \text{ converges to } 0 \}.
$$

For a fixed regular matrix $S$, let $\Sigma$ denote the sequence of matrices $\{S^0, S^1, S^2, \ldots\}$, where $S^v$ denotes the product of $S$ with itself $v$ times. We shall use the notation $A \cdot \Sigma$ to denote the sequence of matrices $\{R^p\}$ where $(R^p x)_n = \Sigma a_{nk}(S^k x)_p$. (Throughout, all sums run from $k = 1$ to $\infty$ unless otherwise noted.) Note that for $T$, the translation matrix, $(T^k x)_p = (T^p x)_k$, so if $S = T$, $A \cdot \Sigma$ reduces to the sequence of ordinary matrix products of $A$ and powers of $T$.

**Theorem 2.** Suppose $A$ is regular. If $A$ is $T$-invariant, then $A \cdot \Sigma$ is $S$-invariant, that is $A \cdot \Sigma(S - I)x$ converges to $0$ for all $x \in m$.

**Proof.** Let $\mathfrak{R} = A \cdot \Sigma$. We show $\mathfrak{R}$ is $S$-invariant. For $x \in m$,

$$(R^p(S - I)x)_n = \Sigma a_{nk}(S^k(S - I)x)_p = \Sigma a_{nk}[(S^{k+1} x) - (S^k x)_p].$$

So

$$
|\lim (R^p(S - I)x_n) | \leq |\Sigma a_{nk-1}(S^k x)_p - \Sigma a_{nk}(S^k x)_p| \\
\leq \Sigma |a_{nk-1} - a_{nk}| ||S^k|| ||x||.
$$

Since $A$ is $T$-invariant, $\Sigma|a_{nk-1} - a_{nk}| \to 0$ as $n \to \infty$ by Shur's Theorem.
Hence \((R^p(S - I)x)_n \rightarrow 0\) as \(n \rightarrow \infty\) uniformly in \(p\). This shows \(A \cdot \Sigma\) is \(S\)-invariant.

Thus we have the following information. If \(A\) is a nonnegative regular matrix which is \(T\)-invariant and \(S\)-invariant, then Theorem 1 states \(S\)-limits exist and Theorem 2 states \(A \cdot \Sigma\) is \(S\)-invariant.

**Theorem 3.** If \(A\) is regular, then \(n_{A\Sigma} \cap m \subset \Theta(S)\).

**Proof.** Suppose \(x \in n_{A\Sigma} \cap m\). Let \(t_n = \{t^v_n, v = 1, 2, \ldots\}\) where \(t^v_n = \Sigma a_{nk}(S^kx)_v\). Thus \(t^v_n \rightarrow 0\) as \(n \rightarrow \infty\) uniformly in \(v = 1, 2, \ldots\), that is \(t_n \rightarrow 0\) in \((m, \|\|_{\infty})\). Let \(L\) be any \(S\)-limit. \(L\) is continuous on \(m\) so \(L(t_n) \rightarrow 0\) as \(n \rightarrow \infty\).

We shall now show \(L(t_n)\) tends to \(L(x)\). Now, for \(n\) fixed we have

\[
\left| t^v_n - \sum_{k=1}^{N} a_{nk}(S^kx)_v \right| \leq \|x\| \sum_{k=N+1}^{\infty} |a_{nk}| \|S^k\| \leq \|x\| \sum_{k=N+1}^{\infty} |a_{nk}|
\]

since \(\|S\| = 1\). For each \(n\), \(\sum_{k=N+1}^{\infty} |a_{nk}| \rightarrow 0\) as \(N \rightarrow \infty\) since \(A\) is regular. Thus \(\sum_{k=1}^{N} a_{nk}(S^kx)_v \rightarrow t^v_n\) as \(N \rightarrow \infty\) uniformly for \(v = 1, 2, \ldots\) for each \(n\). This implies, for an \(S\)-limit \(L\), that

\[
L(t_n) = \lim_{N \rightarrow \infty} \sum_{k=1}^{N} a_{nk} L(S^kx) = \lim_{N \rightarrow \infty} \sum_{k=1}^{N} a_{nk} L(x) = L(x) \sum_{k=1}^{\infty} a_{nk}
\]

since \(L\) is continuous and \(S\)-invariant. Hence \(\lim_{n \rightarrow \infty} L(t_n) = L(x)\) since \(A\) is regular.

Therefore, since \(L(t_n)\) tends to 0 and to \(L(x)\), \(x \in \Theta(S)\), which is the desired result.

**Theorem 4.** The closure of \((S - I)[m]\) in \((m, \|\|)\) is \(\Theta(S)\).

**Proof.** Since an \(S\)-limit \(L\) is \(S\)-invariant, clearly \((S - I)[m] \subset \Theta(S)\). We shall show that every positive linear functional which is zero on \((S - I)[m]\) is also zero on \(\Theta(S)\). (The positive linear functionals are total in the dual of \(\Theta(S)\); see, e.g. [1, Theorem 4, p. 217].)

Suppose \(q\) is a positive linear functional on \(\Theta(S)\) which vanishes on \((S - I)[m]\). We can extend \(q\) to a positive linear functional \(G\) on \(m\) with \(G(\bar{1}) = 1\) (see, e.g., [5, p. 20]). Also \(G((S - I)x) = 0\) for all \(x \in m\), so \(G\) is \(S\)-invariant. Hence \(G\) is an \(S\)-limit, so \(G\) is zero on \(\Theta(S)\). Therefore the closure of \((S - I)[m]\) is \(\Theta(S)\).

**Corollary 1.** \(\overline{bs} = f_0\).

**Proof.** \(bs = \{x: \sup_n \left| \sum_{k=1}^{n} x_k \right| < \infty\}\). It is easy to show \(bs = (T - I)[m]\) where \(T\) is the translation matrix.

\(f_0\) is the space of sequences that are almost convergent to zero which is precisely \(\Theta(T)\).

The following result links \(S\)-limits to \(A \cdot \Sigma\) summability.
Theorem 5. If $A$ is regular and if $A \cdot \Sigma$ is $S$-invariant, then $n_{A \cdot \Sigma} \cap m = \Theta(S)$.

Proof. $A \cdot \Sigma$ is $S$-invariant, so $(S - I)[m] \subseteq n_{A \cdot \Sigma} \cap m$ and $n_{A \cdot \Sigma} \cap m \subseteq \Theta(S)$ from Theorem 3. Since $n_{A \cdot \Sigma} \cap m$ is closed in $m$ and $(S - I)[m] = \Theta(S)$, we have $n_{A \cdot \Sigma} \cap m = \Theta(S)$.

Corollary. If, in addition, $\bar{1} \in c_{A \cdot \Sigma}$, then $c_{A \cdot \Sigma} \cap m = \Lambda(S)$.

We note that $\bar{1} \in c_{A \cdot \Sigma}$ if $A \cdot \Sigma$ is regular or if $A$ is regular and $S$ is positive with all rows adding up to 1. $A$ is said to be compatible to $S$ if $A$ is regular, $A \cdot \Sigma$ is a regular, $S$-invariant sequence of matrices. Thus if $A$ is compatible to $S$, $c_{A \cdot \Sigma} \cap m = \Lambda(S)$. Necessary and sufficient conditions for compatibility can be formulated using Silverman-Toeplitz and Shur Theorems for $\alpha$-summability found in [2].

Comparison theorems. We shall assume $A$ is compatible to $S$. Let $\bar{\Sigma} = A \cdot \Sigma$. Clearly if $B$ were also compatible to $S$, then $c_{A \cdot \Sigma} \cap m \subseteq c_{B \cdot \Sigma} \cap m$. We show that among regular matrices the compatible ones are the “strongest”.

Theorem 6. If $B$ is regular, then $n_{B \cdot \Sigma} \cap m \subseteq n_{A \cdot \Sigma} \cap m$ if, in addition, $1 \in c_{B \cdot \Sigma}$, $c_{B \cdot \Sigma} \cap m \subseteq c_{A \cdot \Sigma} \cap m$.

Proof. Theorem 3 gives that $n_{B \cdot \Sigma} \cap m \subseteq \Theta(S)$ and compatibility gives that $n_{A \cdot \Sigma} \cap m = \Theta(S)$. This gives the desired conclusion.

If $1 \in c_{B \cdot \Sigma}$, since $A \cdot \Sigma$ is regular, we have $c_{B \cdot \Sigma} \cap m \subseteq c_{A \cdot \Sigma}$.

We say a matrix $B$ is strongly $\theta$-regular if $Bx \to l$ whenever $\theta x \to l$. For $\mathcal{S} = C \cdot \Gamma$ where $C$ is a Cesàro matrix and $\Gamma = \{T^n\}$ the sequence of powers of $T$, strongly $\mathcal{S}$-regular corresponds to strongly regular [6]. We have the following generalization of Lorentz’ Theorem.

Theorem 7. $B$ is strongly $\mathcal{S}$-regular on $m$ if and only if $B$ is regular and $S$-invariant (or $\lim_{n \to \infty} \sum \left|(BS)_{nk} - B_{nk}\right| = 0$).

Proof. Suppose $B$ is $\mathcal{S}$-regular on $m$. Then

$$n_B \cap m \supseteq n_{A \cdot \Sigma} \cap m \supseteq (S - I)[m].$$

So $B$ is $S$-invariant. $A \cdot \Sigma$ is regular, so $x \to l$ implies $\mathcal{S}x \to l$, thus $Bx \to l$. $B$ is thus regular.

Conversely, $B$ is $S$-invariant implies $n_B \cap m \supseteq (S - I)[m]$. $n_B \cap m$ is closed, so $n_B \cap m \supseteq \Theta(S)$. $B$ is regular, thus $c_B \cap m \supseteq \Lambda(S)$; that $B$ and $\mathcal{S}$ are consistent follows from $n_B \cap B \supseteq \Theta(S)$.

As a consequence, we have a shortened proof of Lorentz’ characterization of strongly regular matrices [6, Theorem 7].

Corollary. $B$ is strongly regular if and only if $B$ is regular and $\lim_{n \to \infty} \sum \left|b_{nk} - b_{n,k+1}\right| = 0$. 

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Proof. This is the above theorem applied to almost convergence, noting that \((BT)_{nk} = B_{n,k-1}\).

For \(\mathfrak{U} = \{A^n\}\), consider the set \(\mathfrak{U}\) of all matrices \(U\) where the \(n\)th row of \(U\)
is the \(n\)th row of \(A^n\) for some \(v\). Following along the idea of G. M. Petersen [7], it can be shown [2] that \(\mathfrak{U}\) sums exactly those sequences that are summed by every member of \(\mathfrak{U}\) to the same value.

The following generalizes a finding of R. J. Duran [4] for almost convergence.

**Theorem 8.** \(c_s \cap m = \cap \{c_B \cap m : B \text{ is regular and } S\text{-invariant}\}\).

**Proof.** If \(B\) is regular and \(S\)-invariant, it is strongly \(S\)-regular on \(m\), hence \(c_s \cap m \subseteq c_B \cap m\). Also, \(c_s = \cap \{c_{U'} : U \in \mathfrak{U}\}\) where \(\mathfrak{U}\) is described above. \(c_s \subseteq c_U\) are consistent implies that each \(U\) is strongly \(S\)-regular and, hence, regular and \(S\)-invariant. Thus \(\mathfrak{U}\) is a subcollection of all regular and \(S\)-invariant matrices, so \(\cap \{c_{U'} : U \in \mathfrak{U}\} \supseteq \cap \{c_B : B \text{ is regular and } S\text{-invariant}\}\), so \(c_s \cap m = \cap \{c_B \cap m : B \text{ is regular and } S\text{-invariant}\}\).

**REFERENCES**


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