ON THE STRUCTURE OF THE FIXED-POINT SET OF A NONEXPANSIVE MAPPING IN A BANACH SPACE

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Abstract. If $C$ is a closed convex subset of a reflexive, strictly convex Banach space $E$, and $T: C \to E$ is a nonexpansive mapping which has a fixed-point in the interior of $C$, then there exists a nonexpansive mapping $T^*: E \to E$ whose fixed-point set in $C$ is the fixed-point set of $T$.

In this note we investigate the structure of the fixed-point set $F(T)$ of a nonexpansive mapping $T: C \to E$, where $C$ is a closed convex subset of a Banach space $E$ and $T$ does not necessarily map $C$ into itself. It is known that $T$ need not have an extension to a nonexpansive mapping $T^*: E \to \text{cl} T(C)$, (De Figueiredo and Karlovitz [3], Bruck [1]), so the following result is of interest:

Theorem. If $C$ is a closed convex subset of a reflexive strictly convex Banach space $E$, $T: C \to E$ is nonexpansive, and $T$ has a fixed point in the interior of $C$, then there exists a nonexpansive mapping $T^*: E \to E$ whose fixed points in $C$ are exactly the fixed points of $T$.

Before proving the theorem, we establish a variant of Lemma 5 of [2]:

Lemma. If $y$ is a fixed point of $T$ interior to $C$ then there exists a nonexpansive retraction of $E$ onto the cone

$$K(y; F(T)) = \text{cl} \cup \{t \cdot F(T) + (1 - t)y : t > 0\}.$$  

Proof. Let $\delta > 0$ be so small that $B = \{x \in E : \|x - y\| \leq \delta\}$ is contained in $C$. Since $Ty = y$ and $T$ is nonexpansive, $T(B) \subset B$. The restriction $T|_B$ is a nonexpansive mapping of $B$ into itself, so by [2, Theorem 2] there exists a nonexpansive retraction $r_1$ of $B$ onto $F(T|_B) = F(T) \cap B$. For $t > 0$ define

$$B_t = tB + (1-t)y, \quad F_t = t \cdot F(T) \cap B + (1-t)y,$$

and

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For $x \in B_t$, $r_t(x) = y$ for $x \in B_t$. $F(T)$ is convex because $E$ is strictly convex, so for $0 < s < t$, $F_s \subset F_t$ and $B_s \subset B_t$. It is easy to verify that $r_t$ is a retraction of $E$ into $F_t$ which is nonexpansive on $B_t$ (but not on $E$). Evidently $\bigcup \{B_t: t > 0\} = E$ and $\text{cl} \bigcup \{F_t: t > 0\} = K(y; F(T))$. Denote $K(y; F(T))$ by $K$. We shall construct a nonexpansive retraction of $E$ onto $K$ as the limit-in an appropriate product space—of a subnet of $\{r_t: t > 0\}$ (limits taken as $t \to \infty$).

For $x$ in $E$ put $E_x = \{u \in E: \|u - y\| \leq \|x - y\|\}$, $P = \prod_{x \in E} E_x$, give $E_x$ its weak topology, and $P$ the corresponding product topology. By the reflexivity of $E$ and Tychonov’s theorem, $P$ is compact. Evidently $\{r_t: t > 0\}$ is a net in $P$, and therefore has a subnet which converges to some $r$ in $P$.

Given any two $x_1$, $x_2$ in $E$, it follows from the weak lower semicontinuity of the norm, the nonexpansiveness of $r_t$ on $B_t$, and the fact that $x_1$, $x_2 \in B_t$ for sufficiently large $t$, that $\|r(x_1) - r(x_2)\| \leq \|x_1 - x_2\|$. That is, $r$ is nonexpansive. Since $F_s \subset F_t$ for $0 < s < t$, $r_t$ leaves such point of $F_s$ fixed; hence so does $r$. The denseness of $\bigcup F_t$ in $K$ and the continuity of $r$ imply that $r$ fixes each point of $K$. Finally, each $r_t$ maps $E$ into $F_t \subset K$, and since $K$ is weakly closed (being closed and convex) it follows that the range of $r$ is contained in $K$. These three facts—$r$ fixes each point of $K$, the range of $r$, is contained in $K$, and $r$ is nonexpansive—mean $r$ is a nonexpansive retraction of $E$ onto $K$. Q.E.D.

Proof of Theorem. Put $R = \bigcap \{K(y; F(T)): y \in F(T) \cap \text{int } C\}$. By hypothesis, $F(T) \cap \text{int } C \neq \emptyset$, and by the Lemma, there exists a nonexpansive retraction of $E$ onto each $K(y; F(T))$; by [2, Theorem 5], therefore, there exists a nonexpansive retraction of $E$ onto $R$. Let $T^*$ be such a retraction. We claim that $T^*$ satisfies the conclusion of the Theorem. Obviously $F(T) \subset C \cap R = F(T^*|_{C})$. If $F(T) \neq F(T^*|_{C})$, let $x_0 \in C \cap R \setminus F(T)$. We reach a contradiction as follows: let $y_0 \in F(T) \cap \text{int } C$. Since $F(T)$ is closed, the intersection of the line segment $[x_0, y_0]$ with $F(T)$ contains a point $z_0$ closest to $x_0$, $z_0 \neq x_0$ (because $x_0 \not\in F(T)$), $x_0 \in C$, and $y_0 \in \text{int } C$, therefore $z_0 \in \text{int } C$. Choose a point $x \neq z_0$ in $[x_0, z_0]$ which is closer to $z_0$ than to bdry $C$; thus $x \not\in F(T)$. Now let $y$ be the point of $F(T)$ which is closest to $x$ (this exists because $F(T)$ is closed and convex and $E$ is reflexive). Since $x$ is closer to $z_0$ (which is in $F(T)$) than to bdry $C$, $y$ must lie in $\text{int } C$. But $R$ is convex, $x_0$, $z_0 \in R$, and $x \in [x_0, z_0]$, so $x \in R$. In particular,

$$x \in K(y; F(T)).$$

To summarize, there exists a point $x$ such that $x \not\in F(T)$, but for the point $y$ of $F(T)$ closest to $x$, $x \in K(y; F(T))$. This is an obvious impossibility, and establishes $F(T^*|_{C}) = F(T)$. Q.E.D.

Corollary. If, in addition to the hypotheses of the Theorem, $F(T) \subset \text{int } C$, then $F(T^*) = F(T)$. License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
PROOF. \( F(T^*) \) is convex and, by the Theorem, \( F(T^*) \cap C = F(T) \subseteq \text{int } C \); thus \( F(T^*) \subseteq C \), hence \( F(T^*) = F(T) \). Q.E.D.

REFERENCES


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