REPRESENTATIONS OF COMPACT GROUPS
ON TOPOLOGICAL VECTOR SPACES: SOME REMARKS

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Abstract. A standard theorem concerning the decomposition of a representation of a compact group on a Hilbert space $E$ is generalized to the case when $E$ is locally convex and quasi-complete. As a corollary, it is shown that if $E$ is topologically irreducible, then it is finite dimensional.

1. Introduction. Let a compact group $G$ be represented unitarily on a Hilbert space $H$. Then there is a "canonical" decomposition $H = \bigoplus_{\sigma \in \hat{G}} H_{\sigma}$, where the invariant subspaces $H_{\sigma}$ are indexed by the elements $\sigma$ of $\hat{G}$, the set of equivalence classes of irreducible unitary representations of $G$. Each subspace $H_{\sigma}$ is characterized as the smallest subspace of $H$ containing all invariant subspaces $V$ of type $\sigma$ ($V$ is of type $\sigma$ if the representation of $G$ on $V$ is in $\sigma$). Each $H_{\sigma}$ may be expressed (nonuniquely) as a direct sum of invariant subspaces $H_{\sigma,i}$ of type $\sigma$.

We here replace $H$ by a quasi-complete topological vector space $E$. Analogues of the above statements are proven. It is also shown that if $E$ is topologically irreducible, then $E$ has finite dimension.

2. Preliminaries.

2.1. Notation. We let $A$ denote a Hausdorff, locally convex, quasi-complete (closed bounded sets are complete) topological vector space, $G$ a compact topological group which is represented continuously on $A$. Thus there is a continuous map $(g, e) \mapsto g \cdot e : G \times E \to E$ such that each map $g : e \mapsto g \cdot e$ is linear. The notation $(G, A)$ indicates that $G$ is represented continuously on $A$. We use $dg, dg'$, etc. to refer to normalized Haar measure on $G$.

2.2. Definitions. Let $\hat{G}$ be the set of equivalence classes of irreducible unitary representations of $G$ [2, 27.3]. If $\sigma \in \hat{G}$, let $d_{\sigma}$ be the dimension of $V$ for some (hence any) $(G, V) \in \sigma$.

2.3. Definition. A representation $(G, A)$ is topologically irreducible if $A$ contains no proper closed $G$-invariant subspaces except $\{0\}$.

We review the definition and certain basic properties of the weak integral; for more details see [1, Chapter III, §§3–4].

2.4. Let $X$ be a locally compact topological space. Let $K(X) = \{ f : X \to C | f$ is continuous with compact support $\}$, $K(X, E) = \{ f : X \to E | f$ is continuous with compact support $\}$. Let $E'$ be the topological dual of $E$ (continu-
ous linear functionals), $E^\ast$ the algebraic dual of $E'$ (all linear functionals).

Let $\mu$ be a measure on $X$ (a continuous linear functional on $(K(X), \mathcal{T})$, $\mathcal{T}$ = topology of uniform convergence on compact sets). If $f \in K(X, E)$, define the weak integral of $f$ with respect to $\mu$ as follows:

$$\left\langle e', \int_X f(x) \, d\mu(x) \right\rangle = \int_X \langle e', f(x) \rangle \, d\mu(x) \quad (e' \in E').$$

2.5. Theorem. (a) If $f \in K(X, E)$, then $\int_X f(x) \, d\mu(x) \in E$.

(b) If $q$ is a continuous seminorm on $E$, then $q(\int_X f(x) \, d\mu(x)) \leq \int_X q(f(x)) \, d\mu(x)$.

(c) If $Y$ is another locally compact space, $\nu$ a measure on $Y$, and $f \in K(X \times Y, E)$, then

$$\int_{X \times Y} f(x, y) \, d\mu(x) \, d\nu(y) = \int_X \left[ \int_Y f(x, y) \, d\nu(y) \right] \, d\mu(x).$$

For the proofs see [1, Chapter III: §3, No. 3, Corollary 2; §3, No. 2, Proposition 6; §3, No. 2, Proposition 2; §4, No. 1, Remark following Theorem 2].

2.6. Let $M(G)$ be the convolution algebra [2, 19.12] of $G$; denote convolution by $\ast$. The action of $G$ on $E$ extends to an algebra homomorphism $U$ from $M(G)$ to $L(E)$, the continuous linear operators on $E$; the extension is given by $U(\nu) \cdot e = \int_G (g \ast e) \, d\nu(g)$. For a proof of this, see the two paragraphs preceding 4.1.1.2 in [3]; note that (i) $G$ is compact, so all measures on $G$ have compact support; (ii) the proof uses only quasi-completeness, not completeness, of $E$.

2.7. Definitions. Let $\sigma \in \hat{G}$, $(G, V) \in \sigma$. Let $v_1, \ldots, v_n$ ($n = d_\sigma$) be an orthonormal basis in $V$, and let $u_{ij}^\sigma(g) = \langle gv_j, v_i \rangle$ ($g \in G$, $1 \leq i, j \leq n$), where $\langle \cdot, \cdot \rangle$ is the inner product in $V$. Let $L_\sigma(G)$ be the linear span in $C(G)$ of the functions $u_{ij}^\sigma$, and let $\mathcal{S}_\sigma(G)$ be the span in $C(G)$ of $\bigcup_{\sigma \in \hat{G}} \mathcal{S}_\sigma(G)$. Let $\lambda_\sigma(g) = \sum_{i=1}^n u_{ii}^\sigma(g)$ ($g \in G$); $\lambda_\sigma$ is the character of $\sigma$. See [2, 27.5–27.8].

2.8. Theorem. (a) (Orthogonality) $\int_G u_{ij}^\sigma(g)u_{kl}^\tau(g) \, dg = \delta_{ik}\delta_{jl}\delta_{\sigma\tau} (\sigma, \tau \in \hat{G})$.

(b) (Peter-Weyl theorem) $\mathcal{S}_\sigma(G)$ is uniformly dense in $C(G)$.

(c) If $\sigma, \tau \in \hat{G}$ and $\phi \in \mathcal{S}_\sigma(G)$, then $\phi \ast d_\delta\lambda_\sigma = d_\delta\lambda_{\tau} \ast \phi = \phi$.

(d) If $\sigma, \tau \in \hat{G}$, then $\lambda_\sigma \ast \lambda_\tau = \delta_{\tau\sigma} \lambda_\sigma / d_\sigma$.

For the proofs, see [2, 27.19; 27.40; 27.24(ii); 27.24(iii)].

2.9. Definition. Let $(G, E)$ be a representation, $E$ quasi-complete. Say that $(G, E)$ is of type $\sigma$ if for some (hence any) $(G, V) \in \sigma$, there is a linear isomorphism $W: V \to E$ such that $W(g \cdot v) = g \cdot W(v)$. (Thus $W$ intertwines $(G, E)$ and $(G, V)$.) Abusing notation, we write $(G, E) \in \sigma$. 

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3. Results. Recall that a locally convex topology is determined by its set of continuous seminorms.

3.1. Definition. A seminorm \( p \) on \( E \) is \( G \)-invariant if \( p(g \cdot e) = p(e) \) (\( g \in G, e \in E \)).

3.2. Lemma. The topology on \( E \) is determined by the set of continuous \( G \)-invariant seminorms.

Proof. Let \( q \) be a continuous seminorm on \( E \). It suffices to show that there is a continuous \( G \)-invariant seminorm \( p \) such that \( q(e) \leq p(e) \) (\( e \in E \)). To prove this, let \( p(e) = \sup_{g \in G} q(ge) \); \( p \) is an invariant seminorm. To demonstrate continuity of \( p \), let \( e_n \to 0 \) (a seminorm on \( E \) is continuous iff it is continuous at 0). Let \( \epsilon > 0 \) be given, and let \( U_\epsilon = \{ e \in E \mid q(e) < \epsilon \} \). Since \( G \) is compact, it acts equicontinuously on \( E \); let \( V \) be a neighborhood of 0 such that \( G \cdot V \subset U_\epsilon \). If \( n > n_0 \Rightarrow e_n \in V \), then \( n > n_0 \Rightarrow p(e_n) = \sup_{g \in G} q(ge_n) < \epsilon \). So \( p \) is continuous.

Let \( \sigma \in \hat{G} \); we define an operator \( P_\sigma : E \to E \) as follows:

\[
P_\sigma(e) = d_\sigma \int_G \tilde{\lambda}_\sigma(g) (ge) \, dg,
\]

where \( \lambda_\sigma \) is the appropriate character (2.7). By 2.5(a), \( P_\sigma \) is well defined; it is easily seen to be linear.

3.3. Proposition. (a) \( P_\sigma \) is continuous (\( \sigma \in \hat{G} \)).
(b) \( P_\sigma P_\tau = P_{\tau \sigma} = 0 \) (\( \sigma \neq \tau \)); \( P_\sigma^2 = P_\sigma \).
(c) \( gP_\sigma = P_\sigma g \) (\( g \in G \)).

Proof. In the proof, we make free use of 2.5(a)-(d).
(a) Let \( e_n \to 0 \) and suppose \( p \) is a continuous \( G \)-invariant seminorm. Then

\[
p(P_\sigma(e_n)) = d_\sigma \left( \int_G \tilde{\lambda}_\sigma(g) (ge_n) \, dg \right) \leq d_\sigma \int_G |\lambda_\sigma(g)| \, p(ge_n) \, dg \leq d_\sigma^2 p(e_n) \to 0.
\]

By 3.1, \( P_\sigma \) is continuous.
(b) Note that

\[
P_\sigma(P_\tau(e)) = d_\sigma^2 \int_G \tilde{\lambda}_\sigma(g) \, \left( \int_G \tilde{\lambda}_\tau(g') \, (g' \cdot e) \, dg' \right) \, dg
\]

\[
= d_\sigma^2 \int_G \tilde{\lambda}_\sigma(g) \, \tilde{\lambda}_\tau(g') \, (gg' \cdot e) \, dg' \, dg
\]

\[
= d_\sigma^2 \int_G \tilde{g} e \left[ \int_G \tilde{\lambda}_\sigma(g) \, \tilde{\lambda}_\tau(g^{-1}g) \, dg \right]
\]

\[
= d_\sigma^2 \int_G \tilde{g} e \tilde{\lambda}_\sigma(g) \ast \tilde{\lambda}_\tau(g) \, d\tilde{g}.
\]

Since \( \lambda_\sigma \ast \lambda_\tau = \delta_\sigma \lambda_\tau / d_\sigma \), the last integral is zero if \( \sigma \neq \tau \), and is \( d_\sigma \int_G \tilde{g} e \tilde{\lambda}_\sigma(g) \, d\tilde{g} = P_\sigma(e) \) if \( \sigma = \tau \).
(c) We compute:
\[ g \cdot P_\sigma(e) = d_\sigma g \cdot \int_G \lambda_\sigma(g') (g' e) \, dg' = d_\sigma \int_G \lambda_\sigma(g') (g g' e) \, dg' \]
\[ = d_\sigma \int_G \lambda_\sigma(g^{-1} g' e) \, dg' = d_\sigma \int_G \lambda_\sigma(g g^{-1} g' e) \, dg' \]
\[ = d_\sigma \int_G \lambda_\sigma(g') (g' g e) \, dg' = P_\sigma(ge). \]

In 2.6, we defined the algebra homomorphism \( U: M(G) \to L(E) \). Let \( E_\sigma = P_\sigma(E) (\sigma \in \hat{G}) \). From 3.3(b) and (c), we see that \( E_\sigma \cap E_\tau = \{0\} \) \((\sigma \neq \tau)\), and \( G \cdot E_\sigma = E_\sigma \).

3.4. **Proposition.** (a) \( E_\sigma \) is closed.

(b) The weak direct sum \( \bigoplus_{\sigma \in \hat{G}} E_\sigma \) is dense in \( E \).

(c) \( P_\sigma(e) = 0 \) for all \( \sigma \in \hat{G} \) iff \( e = 0 \).

**Proof.** (a) If \( (e_n) \) is a net in \( E_\sigma \) with \( e_n \to e \in E \), then \( P_\sigma(e) = \lim_n P_\sigma(e_n) = \lim_n e_n = e \).

(b) Write \( \Phi_\sigma(G) \) for \( \{ \phi \in \Phi_\sigma(G) \} \). It follows from 2.8(c) that, if \( \phi \in \Phi_\sigma(G) \), then \( \phi \circ d_\sigma \lambda_\sigma = d_\sigma \lambda_\sigma \circ \phi = \phi \). Since \( U \) is a \(*\)-homomorphism, one has \( U(\phi) P_\sigma = P_\sigma U(\phi) = U(\phi) \) (we used the fact that \( U(d_\sigma \lambda_\sigma) = P_\sigma \), which is immediate from the definitions). We now reason as follows. Let \( S = \text{cls} \bigoplus_{\sigma \in \hat{G}} E_\sigma \) and let \( \eta \in E' \) be such that \( \eta|_S = 0 \). Fix \( \sigma \in \hat{G} \) and \( e \in E \); then for all \( \phi \in \Phi_\sigma(G) \), one has \( P_\sigma U(\phi) e \in S \), so
\[ 0 = \langle P_\sigma U(\phi) e, \eta \rangle = \langle U(\phi) e, \eta \rangle = \int_G \langle ge, \eta \rangle \phi(g) \, dg. \]
Let \( \sigma \) vary over \( \hat{G} \); then \( \phi \) may be any element of \( \bigcup_{\sigma \in \hat{G}} \Phi_\sigma(G) \), which has dense linear span in \( C(G) \) (2.8(b)). It follows that \( \langle ge, \eta \rangle = 0 \) for all \( g \in G \). Let \( g = \text{id} \) to see that \( \eta = 0 \). Since \( E \) is locally convex, \( S = E \).

(c) Suppose \( P_\sigma(e) = 0 \) \((\sigma \in \hat{G})\). Then \( 0 = U(\phi) P_\sigma e = U(\phi)e (\phi \in \Phi_\sigma(G)) \).

Repeating part of the proof of (b) shows that \( \langle e, \eta \rangle = 0 \) for each \( \eta \in E' \), so \( e = 0 \).

Recalling 2.7, pick \((G, V) \in \sigma\). Let \( v_1, \ldots, v_n \) \((n = d_\sigma)\) be an orthonormal basis for \( V \), and let \( \{ u_{ij} | 1 \leq i, j \leq n \} \) be the corresponding coordinate functions.

3.5. **Lemma.** Let \( 0 \neq e \in E_\sigma \). Then there exist finitely many invariant subspaces \( (B_j)_{j=1}^l \), \( 1 \leq l \leq d_\sigma \), \((G, B_j) \in \sigma\), such that \( e \in \oplus_{j=1}^l B_j \) (the sum, not the direct sum).

**Proof.** We have \( \lambda_\sigma(g) = \sum_{j=1}^n u_{ij}(g) \). Hence one has \( e = P_\sigma(e) = d_\sigma \sum_{j=1}^l U(\tilde{u}_{ij}) \cdot e \). Let \( J \subset \{ 1, 2, \ldots, n \} \) be those integers such that \( U(\tilde{u}_{ij}) e \neq 0 \). Fixing \( j \in J \), we will show that \( \{ U(\tilde{u}_{ij}) e | 1 \leq i \leq n \} \) spans a subspace \( B_j \) of the desired type.

We first show that if \( g \in G \), then
\[ g \cdot U(\tilde{u}_{ij}) = \sum_{p=1}^j u_{ip}(g) U(\tilde{u}_{pj}). \]
To see this, let \( a \in E, \eta \in E^* \), and let \( g' : E' \to E'' \) be the adjoint of \( g \). Then

\[
\langle g \cdot U(\bar{u}_r)a, \eta \rangle = \langle U(\bar{u}_r)a, g'\eta \rangle = \int_G \langle g'a, g'\eta \rangle \bar{u}_r(g') \, dg' \\
= \int_G \langle gg'a, \eta \rangle \bar{u}_r(g') \, dg' = \int_G \langle g'a, \eta \rangle \bar{u}_r(g^{-1}g') \, dg'.
\]

Now

\[
\bar{u}_r(g^{-1}g') = \sum_{p=1}^n \bar{u}_{r_p}(g^{-1}) \bar{u}_{p_r}(g'),
\]

so the equality continues:

\[
\sum_{p=1}^n u_{pr}(g) \int_G \langle g'a, \eta \rangle \bar{u}_{p_r}(g') \, dg' = \sum_{p=1}^n u_{pr}(g) \langle U(\bar{u}_{p_r}) \cdot a, \eta \rangle.
\]

Comparing first and last terms show that (*) holds.

Recall \( j \in J \) is fixed. Let \( e_i = U(\bar{u}_g) \cdot e \) \((1 \leq i \leq n)\). Since \( U(\bar{u}_g)P = P \cdot U(\bar{u}_g), e_i \in E \). Define \( W : V \to B_j = \text{span}\{e_1, \ldots, e_n\} \) by \( W(v_i) = e_i \) and by linearity. We compute:

\[
W(gv_i) = W\left( \sum_{p=1}^n u_{pi}(g)v_p \right) = \sum_{p=1}^n u_{pi}(g)e_p
\]

by (*). The last term is \( g \cdot e = gW(v_i) \); we conclude that \( B_j \) is \( G \)-invariant and that \( W \) intertwines \((G,V)\) and \((G,B_j)\). Since \( e_j \neq 0 \), \((G,B_j) \in \sigma\).

Apply the above analysis to each \( j \in J \). We obtain subspaces \( B_j \subset E_\sigma \) such that \((G,B_j) \in \sigma\). Any two of these subspaces either coincide or intersect in \( \{0\} \) by irreducibility. Form the sum \( B = \bigoplus_{j \in J} B_j \); then \( e = \sum_{j \in J} U(\bar{u}_g) \cdot e \in B \).

Lemma 3.5 shows that if \( E_\sigma \neq 0 \), then \( E \) contains “many” irreducible subspaces. We can restate this as follows.

3.6. Proposition. If \( E_\sigma \neq 0 \), there is a collection \( (B_i)_{i \in I} \) of \( G \)-invariant subspaces of \( E_\sigma \), with \((G,B_i) \in \sigma\), such that \( \bigoplus_{i \in I} B_i = E_\sigma \) (the weak sum, not its closure!).

Proof. Let \( M \) be the set of all collections \((B_i)_{i \in I'}\) of distinct subspaces of \( E_\sigma \) satisfying \((G,B_i) \in \sigma\); partially order \( M \) by inclusion. By Lemma 3.5, \( M \neq \emptyset \). Clearly we can zornify; let \((B_i)_{i \in I} \) be maximal in \( M \). Since the \( B_i \) are distinct and irreducible, \( B_i \cap B_j = \{0\} \) \((i \neq j)\). We now define \( B = \bigoplus_{i \in I} B_i \). If \( e \in E_\sigma, e \not\in B \), perform the construction of 3.5 to obtain subspaces \( C_j, 1 \leq j \leq l \), such that \((G,C_j) \in \sigma \). At least one \( C_{j_0} \) is not in \((B_i)_{i \in I} \), for otherwise \( e \in \bigoplus_{j=j_1} C_j \subset \bigoplus_{i \in I} B_i \). Add \( C_{j_0} \) to \((B_i)_{i \in I} \) to obtain a contradiction.
3.7. Proposition. $E_{a}$ is the smallest closed subspace of $E$ containing all $G$-invariant subspaces $B$ such that $(G, B) \in \sigma$.

Proof. Let $B \subset E$, $(G, B) \in \sigma$, and let $V, v_1, \ldots, v_n, u_{ij}$ be as in 3.5. Let $W : V \to B$ be intertwining, $e_i = W(v_i)$. Then

$$P_{a}(E_{i}) = d_{o}\int_{G} (ge_{i})(\lambda_{a}(g)) \, dg = d_{o}\int_{G} g \cdot W(v_{i}) \lambda_{a}(g) \, dg$$

$$= d_{o}\int_{G} W(gv_{i}) \lambda_{a}(g) \, dg = W\left(\int_{G} (gv_{i}) \lambda_{a}(g) \, dg\right).$$

The last step uses 2.5(c) and the fact that $W$ is continuous as a map from $V$ to $E$. The equality continues:

$$= W\left[d_{o} \sum_{p=1}^{n} u_{pi}(g)v_{p}\right] = W\left[d_{o} \sum_{p=1}^{n} u_{pi}(g)u_{ri}(g) \, dg\right] = W(\delta_{i,0}) = e_{i}.$$

We have used the orthogonality results for the $u_{ij}$'s (2.8(a)). This shows that $P_{a}e_{i} = e_{i}$, i.e., $e_{i} \in E$. Thus $B \subset E_{a}$, so $E_{a}$ contains all invariant subspaces such that $(G, B) \in \sigma$. Lemma 3.5 shows that $E_{a}$ is the smallest such.

3.8. Remark. The proofs of 3.4, 3.5, and 3.7 are straightforward generalizations of portions of the proof of 27.44 in [2].

3.9. Theorem. Let $E \neq \{0\}$ be topologically irreducible. Then $(G, E) \in \sigma$ for some $\sigma \in \hat{G}$. In particular, $E$ has finite dimension.

Proof. Since $E \neq \{0\}$, there exists $\sigma \in \hat{G}$ such that $P_{\sigma} \neq 0$ (3.4(c)). Consider one such $\sigma$; the space $E_{a}$ is invariant and closed, hence $E_{a} = E$. Let $e \in E_{a}$; by 3.5, $e \in \bigoplus_{j=1}^{l} B_{j}$, where each $(G, B_{j}) \in \sigma$. By finite dimensionality $+\bigoplus_{j=1}^{l} B_{j}$ is closed; it follows easily that $l = 1$ and that $(G, E) \in \sigma$.

3.10. Corollary. Let $E' \subset E$ be topologically irreducible. Then $E' \subset E_{a}$ for some $\sigma \in \hat{G}$.

Proof. Since $(G, E') \in \sigma$ for some $\sigma$ (3.9), $E' \subset E_{a}$ (3.7).

References


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