PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 61, Number 1, November 1976

REPRESENTATIONS OF COMPACT GROUPS ON TOPOLOGICAL VECTOR SPACES: SOME REMARKS

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ABSTRACT. A standard theorem concerning the decomposition of a representation of a compact group on a Hilbert space E is generalized to the case when E is locally convex and quasi-complete. As a corollary, it is shown that if E is topologically irreducible, then it is finite dimensional.

1. **Introduction.** Let a compact group G be represented unitarily on a Hilbert space H. Then there is a "canonical" decomposition $H = \bigoplus_{\sigma \in \hat{G}} H_\sigma$, where the invariant subspaces H_σ are indexed by the elements σ of $\hat{G}$, the set of equivalence classes of irreducible unitary representations of G. Each subspace H_σ is characterized as the smallest subspace of H containing all invariant subspaces V of type σ (V is of *type* σ if the representation of G on V is in σ). Each H_σ may be expressed (nonuniquely) as a direct sum of invariant subspaces $H_{\sigma,i}$ of type σ.

We here replace H by a quasi-complete topological vector space E. Analogues of the above statements are proven. It is also shown that if E is topologically irreducible, then E has finite dimension.

2. **Preliminaries.**

2.1. *Notation*. We let E denote a Hausdorff, locally convex, quasi-complete (closed bounded sets are complete) topological vector space, G a compact topological group which is represented continuously on E. Thus there is a continuous map $(g, e) \to g \cdot e$: $G \times E \to E$ such that each map g: $e \to g \cdot e$ is linear. The notation (G, E) indicates that G is represented continuously on E. We use dg, dg', etc. to refer to normalized Haar measure on G.

2.2. DEFINITIONS. Let $\hat{G}$ be the set of equivalence classes of irreducible *unitary* representations of G [2, 27.3]. If $\sigma \in \hat{G}$, let d_σ be the dimension of V for some (hence any) $(G, V) \in \sigma$.

2.3. DEFINITION. A representation (G, E) is *topologically irreducible* if E contains no proper closed G-invariant subspaces except $\{0\}$.

We review the definition and certain basic properties of the weak integral; for more details see [1, Chapter III, §§3–4].

2.4. Let X be a locally compact topological space. Let $K(X) = \{f\colon X \to C \mid f$ is continuous with compact support$\}$, $K(X, E) = \{f\colon X \to E \mid f$ is continuous with compact support$\}$. Let E' be the *topological* dual of E (*continu-*

Received by the editors January 1, 1976.
AMS (MOS) subject classifications (1970). Primary 22C05, 28A45.

ous linear functionals), E'^* the *algebraic* dual of E' (*all* linear functionals). Let μ be a measure on X (a continuous linear functional on $(K(X), \mathfrak{T})$, $\mathfrak{T}$ = topology of uniform convergence on compact sets). If $f \in K(X, E)$, define the *weak integral of f with respect to μ* as follows:

$$\left\langle e', \int_X f(x)\, d\mu(x) \right\rangle = \int_X \langle e', f(x) \rangle\, d\mu(x) \qquad (e' \in E').$$

2.5. THEOREM. (a) *If $f \in K(X, E)$, then $\int_X f(x)\, d\mu(x) \in E$.*

(b) *If q is a continuous seminorm on E, then $q(\int_X f(x)\, d\mu(x)) \leqslant \int_X q(f(x))\, d\mu(x)$.*

(c) *If Y is another locally compact space, ν a measure on Y, and*

$$f \in K(X \times Y, E),$$

then

$$\int_{X \times Y} f(x, y)\, d\mu(x)\, d\nu(y) = \int_X d\mu(x) \left[\int_Y f(x, y)\, d\nu(y) \right]$$
$$= \int_Y d\nu(y) \left[\int_X f(x, y)\, d\mu(x) \right].$$

For the proofs see [1, Chapter III: §3, No. 3, Corollary 2; §3, No. 2, Proposition 6; §3, No. 2, Proposition 2; §4, No. 1, Remark following Theorem 2].

2.6. Let $M(G)$ be the convolution algebra [2, 19.12] of G; denote convolution by $*$. The action of G on E extends to an algebra homomorphism U from $M(G)$ to $L(E)$, the continuous linear operators on E; the extension is given by $U(\nu) \cdot e = \int_G (g \cdot e)\, d\nu(g)$. For a proof of this, see the two paragraphs preceding 4.1.1.2 in [3]; note that (i) G is compact, so all measures on G have compact support; (ii) the proof uses only quasi-completeness, not completeness, of E.

2.7. DEFINITIONS. Let $\sigma \in \hat{G}$, $(G, V) \in \sigma$. Let $v_1, \ldots, v_n$ $(n = d_\sigma)$ be an orthonormal basis in V, and let $u_{ij}^\sigma(g) = \langle g v_j, v_i \rangle$ $(g \in G,\ 1 \leqslant i, j \leqslant n)$, where $\langle\ ,\ \rangle$ is the inner product in V. Let $\mathfrak{T}_\sigma(G)$ be the linear span in $C(G)$ of the functions u_{ij}^σ, and let $\mathfrak{T}(G)$ be the span in $C(G)$ of $\bigcup_{\sigma \in \hat{G}} \mathfrak{T}_\sigma(G)$. Let $\lambda_\sigma(g) = \sum_{i=1}^n u_{ii}^\sigma(g)$ $(g \in G)$; λ_σ is the *character* of σ. See [2, 27.5–27.8].

2.8. THEOREM. (a) (*Orthogonality*) $\int_G u_{ij}^\sigma(g) u_{kl}^\tau(g)\, dg = \delta_{ik}\delta_{jl}\delta_{\sigma\tau}$ $(\sigma, \tau \in \hat{G})$.

(b) (*Peter-Weyl theorem*) *$\mathfrak{T}(G)$ is uniformly dense in $C(G)$.*

(c) *If $\sigma, \tau \in \hat{G}$ and $\phi \in \mathfrak{T}_\sigma(G)$, then $\phi * d_\sigma \lambda_\sigma = d_\sigma \lambda_\sigma * \phi = \phi$.*

(d) *If $\sigma, \tau \in \hat{G}$, then $\lambda_\sigma * \lambda_\tau = \delta_{\sigma\tau} \lambda_\sigma / d_\sigma$.*

For the proofs, see [2, 27.19; 27.40; 27.24(ii); 27.24(iii)].

2.9. DEFINITION. Let (G, E) be a representation, E quasi-complete. Say that (G, E) is *of type σ* if for some (hence any) $(G, V) \in \sigma$, there is a linear isomorphism $W: V \to E$ such that $W(g \cdot v) = g \cdot W(v)$. (Thus W *intertwines* (G, E) and (G, V).) Abusing notation, we write $(G, E) \in \sigma$.

3. **Results.** Recall that a locally convex topology is determined by its set of continuous seminorms.

3.1. DEFINITION. A seminorm p on E is *G-invariant* if $p(g \cdot e) = p(e)$ $(g \in G, e \in E)$.

3.2. LEMMA. *The topology on E is determined by the set of continuous G-invariant seminorms.*

PROOF. Let q be a continuous seminorm on E. It suffices to show that there is a continuous G-invariant seminorm p such that $q(e) \leqslant p(e)$ $(e \in E)$. To prove this, let $p(e) = \sup_{g \in G} q(ge)$; p is an invariant seminorm. To demonstrate continuity of p, let $e_n \to 0$ (a seminorm on E is continuous iff it is continuous at 0). Let $\varepsilon > 0$ be given, and let $U_\varepsilon = \{e \in E | q(e) < \varepsilon\}$. Since G is compact, it acts equicontinuously on E; let V be a neighborhood of 0 such that $G \cdot V \subset U_\varepsilon$. If $n > n_0 \Rightarrow e_n \in V$, then $n \geqslant n_0 \Rightarrow p(e_n) = \sup_{g \in G} q(ge_n) < \varepsilon$. So p is continuous.

Let $\sigma \in \hat{G}$; we define an operator P_σ: $E \to E$ as follows:

$$P_\sigma(e) = d_\sigma \int_G \overline{\lambda_\sigma(g)}\,(ge)\,dg,$$

where λ_σ is the appropriate character (2.7). By 2.5(a), P_σ is well defined; it is easily seen to be linear.

3.3. PROPOSITION. (a) *P_σ is continuous* $(\sigma \in \hat{G})$.
(b) $P_\sigma P_\tau = P_\tau P_\sigma = 0$ $(\sigma \neq \tau)$; $P_\sigma^2 = P_\sigma$.
(c) $gP_\sigma = P_\sigma g$ $(g \in G)$.

PROOF. In the proof, we make free use of 2.5(a)–(d).

(a) Let $e_n \to 0$ and suppose p is a continuous G-invariant seminorm. Then

$$p(P_\sigma(e_n)) = d_\sigma p\left(\int_G \overline{\lambda_\sigma(g)}\,(ge_n)\,dg\right) \leqslant d_\sigma \int_G \left|\overline{\lambda_\sigma(g)}\right| p(ge_n)\,dg \leqslant d_\sigma^2 p(e_n) \to 0.$$

By 3.1, P_σ is continuous.

(b) Note that

$$\begin{aligned} P_\sigma(P_\tau(e)) &= d_\sigma^2 \int_G \overline{\lambda_\sigma(g)}\, g \cdot \left[\int_G \overline{\lambda_\tau(g')}\,(g'e)\,dg'\right] dg \\ &= d_\sigma^2 \int_G \int_G \overline{\lambda_\tau(g)}\; \overline{\lambda_\sigma(g')}\,(gg'e)\,dg'\,dg \\ &= d_\sigma^2 \int_G \tilde{g}e\left[\int_G \overline{\lambda_\sigma(g)}\; \overline{\lambda_\tau(g^{-1}\tilde{g})}\,dg\right] \\ &= d_\sigma^2 \int_G (\tilde{g}e)\,\overline{\lambda_\sigma * \lambda_\tau(\tilde{g})}\,d\tilde{g}. \end{aligned}$$

Since $\lambda_\sigma * \lambda_\tau = \delta_{\sigma\tau}\lambda_\sigma / d_\sigma$, the last integral is zero if $\sigma \neq \tau$, and is $d_\sigma \int_G (\tilde{g}e)\overline{\lambda_\sigma(\tilde{g})}\,d\tilde{g} = P_\sigma(e)$ if $\sigma = \tau$.

(c) We compute:

$$g \cdot P_\sigma(e) = d_\sigma g \cdot \int_G \overline{\lambda_\sigma(g')}\,(g'e)\,dg' = d_\sigma \int_G \overline{\lambda_\sigma(g')}\,(gg'e)\,dg'$$

$$= d_\sigma \int_G \overline{\lambda_\sigma(g^{-1}\tilde{g})}\,(\tilde{g}e)\,d\tilde{g} = d_\sigma \int_G \overline{\lambda_\sigma(\tilde{g}g^{-1})}\,(\tilde{g}e)\,d\tilde{g}$$

$$= d_\sigma \int_G \overline{\lambda_\sigma(g')}\,(g'ge)\,dg' = P_\sigma(ge).$$

In 2.6, we defined the algebra homomorphism $U\colon M(G) \to L(E)$.

Let $E_\sigma = P_\sigma(E)$ ($\sigma \in \hat{G}$). From 3.3(b) and (c), we see that $E_\sigma \cap E_\tau = \{0\}$ ($\sigma \neq \tau$), and $G \cdot E_\sigma = E_\sigma$.

3.4. Proposition. (a) *E_σ is closed.*

(b) *The weak direct sum $\bigoplus_{\sigma \in \hat{G}} E_\sigma$ is dense in E.*

(c) *$P_\sigma(e) = 0$ for all $\sigma \in \hat{G}$ iff $e = 0$.*

Proof. (a) If (e_n) is a net in E_σ with $e_n \to e \in E$, then $P_\sigma(e) = \lim_n P_\sigma(e_n) = \lim_n e_n = e$.

(b) Write $\overline{\mathfrak{T}_\sigma(G)}$ for $\{\bar{\phi} | \phi \in \mathfrak{T}_\sigma(G)\}$. It follows from 2.8(c) that, if $\phi \in \overline{\mathfrak{T}_\sigma(G)}$, then $\phi * d_\sigma \bar{\lambda}_\sigma = d_\sigma \bar{\lambda}_\sigma * \phi = \phi$. Since U is a *-homomorphism, one has $U(\phi)P_\sigma = P_\sigma U(\phi) = U(\phi)$ (we used the fact that $U(d_\sigma \bar{\lambda}_\sigma) = P_\sigma$, which is immediate from the definitions). We now reason as follows. Let $S = \operatorname{cls} \bigoplus_{\sigma \in \hat{G}} E_\sigma$, and let $\eta \in E'$ be such that $\eta|_S = 0$. Fix $\sigma \in \hat{G}$ and $e \in E$; then for all $\phi \in \overline{\mathfrak{T}_\sigma(G)}$, one has $P_\sigma U(\phi)e \in S$, so

$$0 = \langle P_\sigma U(\phi) \cdot e, \eta \rangle = \langle U(\phi)e, \eta \rangle = \int_G \langle ge, \eta \rangle \phi(g)\,dg.$$

Let σ vary over $\hat{G}$; then ϕ may be any element of $\bigcup_{\sigma \in \hat{G}} \overline{\mathfrak{T}_\sigma(G)}$, which has dense linear span in $C(G)$ (2.8(b)). It follows that $\langle ge, \eta \rangle = 0$ for all $g \in G$. Let $g = \mathrm{id}$ to see that $\eta = 0$. Since E is locally convex, $S = E$.

(c) Suppose $P_\sigma(e) = 0$ ($\sigma \in \hat{G}$). Then $0 = U(\phi)P_\sigma e = U(\phi)e$ ($\phi \in \overline{\mathfrak{T}_\sigma(G)}$). Repeating part of the proof of (b) shows that $\langle e, \eta \rangle = 0$ for each $\eta \in E^*$, so $e = 0$.

Recalling 2.7, pick $(G, V) \in \sigma$. Let $v_1, \ldots, v_n$ ($n = d_\sigma$) be an orthonormal basis for V, and let $\{u_{ij} | 1 \leqslant i, j \leqslant n\}$ be the corresponding coordinate functions.

3.5. Lemma. *Let $0 \neq e \in E_\sigma$. Then there exist finitely many invariant subspaces $(B_j)_{j=1}^{l}$, $1 \leqslant l \leqslant d_\sigma$, $(G, B_j) \in \sigma$, such that $e \in +_{j=1}^{l} B_j$ (the sum, not the direct sum).*

Proof. We have $\lambda_\sigma(g) = \sum_{j=1}^n u_{jj}(g)$. Hence one has $e = P_\sigma(e) = d_\sigma \sum_{j=1}^n U(\bar{u}_{jj}) \cdot e$. Let $J \subset \{1, 2, \ldots, n\}$ be those integers such that $U(\bar{u}_{jj})e \neq 0$. Fixing $j \in J$, we will show that $\{U(\bar{u}_{ij})e | 1 \leqslant i \leqslant n\}$ spans a subspace B_j of the desired type.

We first show that if $g \in G$, then

$$(*) \qquad g \cdot U(\bar{u}_{rs}) = \sum_{p=1}^{j} u_{pr}(g) U(\bar{u}_{ps}).$$

To see this, let $a \in E$, $\eta \in E^*$, and let $g'\colon E' \to E''$ be the adjoint of g. Then

$$\begin{aligned}\langle g \cdot U(\bar{u}_{rs})a, \eta\rangle &= \langle U(\bar{u}_{rs})a, g'\eta\rangle = \int_G \langle g'a, g'\eta\rangle \bar{u}_{rs}(g')\,dg' \\ &= \int_G \langle gg'a, \eta\rangle \bar{u}_{rs}(g')\,dg' = \int_G \langle g'a, \eta\rangle \bar{u}_{rs}(g^{-1}g')\,dg'.\end{aligned}$$

Now

$$\bar{u}_{rs}(g^{-1}g') = \sum_{p=1}^{n} \bar{u}_{rp}(g^{-1})\bar{u}_{ps}(g'),$$

so the equality continues:

$$\sum_{p=1}^{n} u_{pr}(g) \int_G \langle g'a, \eta\rangle \bar{u}_{ps}(g')\,dg' = \sum_{p=1}^{n} u_{pr}(g)\langle U(\bar{u}_{ps}) \cdot a, \eta\rangle.$$

Comparing first and last terms show that $(*)$ holds.

Recall $j \in J$ is fixed. Let $e_i = U(\bar{u}_{ij}) \cdot e$ $(1 \leqslant i \leqslant n)$. Since $U(\bar{u}_{ij})P = P_\sigma U(\bar{u}_{ij})$, $e_i \in E$. Define $W\colon V \to B_j = \operatorname{span}\{e_1, \ldots, e_n\}$ by $W(v_i) = e_i$ and by linearity. We compute:

$$\begin{aligned}W(gv_i) &= W\left(\sum_{p=1}^{n} u_{pi}(g)v_p\right) = \sum_{p=1}^{n} u_{pi}(g)e_p \\ &= \sum_{p=1}^{n} u_{pi}(g)U(\bar{u}_{pj}) \cdot e = g \cdot U(\bar{u}_{ij})(e)\end{aligned}$$

by $(*)$. The last term is $g \cdot e_i = gW(v_i)$; we conclude that B_j is G-invariant and that W intertwines (G, V) and (G, B_j). Since $e_j \neq 0$, $(G, B_j) \in \sigma$.

Apply the above analysis to each $j \in J$. We obtain subspaces $B_j \subset E_\sigma$ such that $(G, B_j) \in \sigma$. Any two of these subspaces either coincide or intersect in $\{0\}$ by irreducibility. Form the sum $B = +_{j \in J} B_j$; then $e = \sum_{j \in J} U(\bar{u}_{jj}) \cdot e \in B$.

Lemma 3.5 shows that if $E_\sigma \neq 0$, then E contains "many" irreducible subspaces. We can restate this as follows.

3.6. PROPOSITION. *If $E_\sigma \neq 0$, there is a collection $(B_i)_{i \in I}$ of G-invariant subspaces of E_σ, with $(G, B_i) \in \sigma$, such that $+_{i \in I} B_i = E_\sigma$ (the weak sum, not its closure!).*

PROOF. Let M be the set of all collections $(B_i')_{i \in I'}$ of distinct subspaces of E_σ satisfying $(G, B_i') \in \sigma$; partially order M by inclusion. By Lemma 3.5, $M \neq \Phi$. Clearly we can zornify; let $(B_i)_{i \in I}$ be maximal in M. Since the B_i are distinct and irreducible, $B_i \cap B_j = \{0\}$ $(i \neq j)$. We now define $B = +_{i \in I} B_i$. If $e \in E_\sigma$, $e \notin B$, perform the construction of 3.5 to obtain subspaces C_j, $1 \leqslant j \leqslant l$, such that $(G, C_j) \in \sigma$. At least one C_{j_0} is not in $(B_i)_{i \in I}$, for otherwise $e \in +_{j=1}^{l} C_j \subset +_{i \in I} B_i$. Add C_{j_0} to $(B_i)_{i \in I}$ to obtain a contradiction.

3.7. PROPOSITION. *E_σ is the smallest closed subspace of E containing all G-invariant subspaces B such that $(G, B) \in \sigma$.*

PROOF. Let $B \subset E$, $(G, B) \in \sigma$, and let $V, v_1, \ldots, v_n, u_{ij}$ be as in 3.5. Let $W: V \to B$ be intertwining, $e_i = W(v_i)$. Then

$$P_\sigma(E_i) = d_\sigma \int_G (ge_i)\overline{(\lambda_\sigma(g))}\, dg = d_\sigma \int_G g \cdot W(v_i)\overline{\lambda_\sigma(g)}\, dg$$

$$= d_\sigma \int_G W(gv_i)\overline{\lambda_\sigma(g)}\, dg = W\left(\int_G (gv_i)\overline{\lambda_\sigma(g)}\, dg\right).$$

The last step uses 2.5(c) and the fact that W is continuous as a map from V to E. The equality continues:

$$= W\left[d_\sigma \int_G \left(\sum_{p=1}^n u_{pi}(g)v_p\right) \cdot \left(\sum_{r=1}^n \overline{u_{rr}(g)}\right) dg\right]$$

$$= W\left[d_\sigma \sum_{p=1}^n \left(\int_G \sum_{r=1}^n u_{pi}(g)\overline{u_{rr}(g)}\, dg\right) v_p\right]$$

$$= W\left[d_\sigma \sum_{p=1}^n \frac{1}{d_\sigma}\delta_{pi}v_p\right] = W(v_i) = e_i.$$

We have used the orthogonality results for the u_{ij}'s (2.8(a)). This shows that $P_\sigma e_i = e_i$; i.e., $e_i \in E$. Thus $B \subset E_\sigma$, so E_σ contains all invariant subspaces such that $(G, B) \in \sigma$. Lemma 3.5 shows that E_σ is the smallest such.

3.8. REMARK. The proofs of 3.4, 3.5, and 3.7 are straightforward generalizations of portions of the proof of 27.44 in [2].

3.9. THEOREM. *Let $E \neq \{0\}$ be topologically irreducible. Then $(G, E) \in \sigma$ for some $\sigma \in \hat{G}$. In particular, E has finite dimension.*

PROOF. Since $E \neq \{0\}$, there exists $\sigma \in \hat{G}$ such that $P_\sigma \neq 0$ (3.4(c)). Consider one such σ; the space E_σ is invariant and closed, hence $E_\sigma = E$. Let $e \in E_\sigma$; by 3.5, $e \in \bigoplus_{j=1}^l B_j$, where each $(G, B_j) \in \sigma$. By finite dimensionality $+_{j=1}^l B_j$ is closed; it follows easily that $l = 1$ and that $(G, E) \in \sigma$.

3.10. COROLLARY. *Let $E' \subset E$ be topologically irreducible. Then $E' \subset E_\sigma$ for some $\sigma \in \hat{G}$.*

PROOF. Since $(G, E') \in \sigma$ for some σ (3.9), $E' \subset E_\sigma$ (3.7).

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