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NEGATIVE TANGENT BUNDLES AND HYPERBOLIC MANIFOLDS

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ABSTRACT. We construct a family of algebraic manifolds which are hyperbolic in the sense of Kobayashi, but whose tangent bundles are not negative.

I. Introduction.

DEFINITION 1. Let M be a complex manifold. The differential Kobayashi metric K_M on M is defined as follows:

$$K_M\colon T(M) \to R^+ \cup \{0\}.$$

Let $(z, v) \in T(M)$; then

$$K_M(z,v) = \inf_f \{|t| \, | df(t) = v, f \in \operatorname{Hol}(D, M), f(0) = z,$$
$$t \text{ is a tangent vector of } D \text{ at the origin}\},$$

where D denotes the unit disc, Hol (D, M) is the family of holomorphic mappings from D to M, and $|t|$ is just the Euclidean norm at the origin of D.

DEFINITION 2. M is said to be hyperbolic iff K_M is nontrivial everywhere on $T(M)$, that is $K_M(z,v) = 0 \leftrightarrow v = 0$, for all $z \in M$ (see Definition 3).

Let M be a compact complex manifold.

DEFINITION 3. The tangent bundle $T(M)$ of M is said to be negative iff $T(M)$ is a strongly pseudo-convex manifold (i.e. admitting a strongly pseudo-convex exhaustion function) whose only compact complex analytic subvariety is the zero section.

A different way to define the negative tangent bundle is: $T(M)$ is negative iff $T^*(M)$ (the cotangent bundle) is ample. (See [5] for the definition of ampleness.)

In [4] S. Kobayashi gave a differential geometric proof of the following

THEOREM. *Let M be a compact complex manifold whose tangent bundle is negative; then M must be hyperbolic.*

A counterexample to the converse of this theorem did not seem to be known. As a consequence of recent results of Brody and Green [1], one can

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obtain such an example from their construction. The aim of this note is to point out this interesting fact.

II. **A family of algebraic hyperbolic manifolds whose tangent bundles are not negative**. In [1], Brody and Green obtained the following remarkable result. (They proved a more general result, but the following description is sufficient for our purpose.)

THEOREM (BRODY AND GREEN). *Let* $W_d = \{W_0^d + W_1^d + W_2^d + W_3^d = 0\} \subset \mathbf{C}P_3$, *where* $d \geqslant 50$, *and is even.*

Let us consider a family of hypersurfaces in $\mathbf{C}P_3$ *given by the following construction*:

$$V_\varepsilon = \{Z_0^d + Z_1^d + Z_2^d + Z_3^d + (\varepsilon z_0 z_1)^{d/2} + \varepsilon(z_0 z_2)^{d/2} = 0\},$$

for ε *sufficiently small.*

Then those surfaces in V_ε *with* $\varepsilon \neq 0$ *are hyperbolic.*

The proof of this theorem depends heavily on the thesis of Robert Brody: **C**-*hyperbolic is equivalent to Kobayashi hyperbolic for compact complex manifold* (Harvard, 1975).

We *claim* that any member of V_ε cannot carry negative tangent bundle. The crucial observation is the following

LEMMA. *Any hypersurface given by homogeneous polynomial of degree* $\geqslant 3$ *in* $\mathbf{C}P_3$ *cannot carry negative tangent bundle.*

(As a matter of fact, this result is true for any hypersurface given by homogeneous polynomials in $\mathbf{C}P_n$ for $n \geqslant 3$, but we just need this lemma to prove our counterexample.

PROOF. We need a result of Kleiman [5], which can be stated as follows.

THEOREM. *Let* M *be any nonsingular algebraic surface, and* E *any ample vector bundle* E *of rank* 2 *over* M. *We let* d_1, d_2 *be the first and second Chern classes of* E; *then the following inequality holds*:

$$(d_1^2 - d_2)(M) > 0.$$

In our case, we take $T^*(M)$ to be E.

Note. From our second definition of negative tangent bundle, $T(M)$ is negative $\leftrightarrow T^*(M)$ is ample.

We let C_1, C_2 be the first and second Chern classes of $T(M)$; then we have the following relations:

$$d_1 = -C_1, \qquad d_2 = (-1)^2 C_2.$$

Note. $\dim_{\mathbf{C}} M = 2$.

In order to prove our lemma, one has to show that if M is the hypersurface in our lemma, then $(C_1^2 - C_2)(M) < 0$.

Let h be the first Chern class of the hyperplane bundle H of $\mathbf{C}P_3$. By a formula in the appendix of [2], one obtains the following identity:

$$(1 + h)^4 = (1 + C_1 + C_2)(1 + d \cdot h)$$

(recall d is the degree of M); then we have

$$C_1 = (4 - d) \cdot h, \qquad C_2 = (6 - 4d + d^2) \cdot h^2.$$

We can easily show

$$(C_1^2 - C_2) = (10 - 4d) \cdot h^2,$$

so that we can conclude

$$(C_1^2 - C_2)(M) < 0 \quad \text{if } d \geqslant 3.$$

This completes the proof of the lemma. Q.E.D.

Now we are in the final touch of our assertion. If $\dim_{\mathbf{C}} M = 2$, one can see $C_1^2(M)$, $C_2(M)$ are topological invariance. (One way to see this fact is the following: $(C_1^2 - 2C_2)(M)$ is the signature of M if $\dim_{\mathbf{C}} M = 2$, and $C_2(M)$ is just the Euler characteristic. Both the signature and Euler characteristic are topological invariance, hence imply $C_1^2(M)$ is also a topological invariance.)

When $\varepsilon = 0$, the hypersurface is only the locus of the following homogeneous polynomial:

$$M_0 = \{Z_0^d + Z_1^d + Z_2^d + Z_3^d = 0\}, \qquad d \geqslant 50.$$

We also observe any member of V_ε is diffeomorphic to M_0, however by the proof of our lemma, $(C_1^2 - C_2^2)(M_0)$ is always less than zero (if $d \geqslant 50$), and this number is a topological invariance from previous argument. Therefore we can easily see that any member of V_ε cannot carry a negative tangent bundle. This completes the whole proof.

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