A SUFFICIENT CONDITION FOR HYPERINVARIANCE

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Abstract. A linear transformation on a finite-dimensional complex linear space has the property that all of its invariant subspaces are hyperinvariant if and only if its lattice of invariant subspaces is distributive [1]. It is shown that an operator on a complex Hilbert space has this property if its lattice of invariant subspaces satisfies a certain distributivity condition.

1. Preliminaries. Throughout this paper \( H \) will denote an arbitrary complex Hilbert space. All operators are bounded and all subspaces are closed. By a subspace lattice on \( H \) is meant a family of subspaces of \( H \) which is closed under the formation of arbitrary intersections and arbitrary closed linear spans and which contains the zero subspace \((0)\) and \( H \). The family of subspaces invariant under an operator \( T \) is denoted by \( \text{Lat } T \). This is a subspace lattice as is the family of subspaces invariant under every operator commuting with \( T \) which we denote by \( \text{Hyperlat } T \). The elements of \( \text{Hyperlat } T \) are called the hyperinvariant subspaces of \( T \). Clearly \( \text{Hyperlat } T \subseteq \text{Lat } T \).

A subspace lattice \( \mathcal{T} \) is called commutative if for every pair of subspaces \( M, N \in \mathcal{T} \) the corresponding projections \( P_M \) and \( P_N \) commute. Let \( L \) be an abstract lattice. We say that \( L \) is

(i) distributive if
\[
(a \lor (b \land c) = (a \lor b) \land (a \lor c) \quad (a, b, c \in L)
\]
and its dual statement holds identically;

(ii) \( \sigma \)-infinitely meet distributive if \( L \) is \( \sigma \)-complete and
\[
a \lor \{\land b_n: n \geq 1\} = \land \{a \lor b_n: n \geq 1\} \quad (a, b_n \in L)
\]
holds identically in \( L \).

That the dual equation defining distributivity are equivalent to each other is an elementary result of lattice theory.

2. A sufficient condition for hyperinvariance. The key to the sufficient condition is a result of Sarason and the following lattice-theoretic result.

Proposition 2.1. If \( \mathcal{T} \) is an abstract \( \sigma \)-infinitely meet distributive lattice and \( \theta: \mathcal{T} \rightarrow \mathcal{T} \) is a lattice automorphism with the properties

(I) \( a \leq \theta(a) \lor \theta^{-1}(a) \quad (a \in \mathcal{T}) \);

(II) \( a, \theta(a) \) comparable implies \( a = \theta(a) \),

then \( \theta \) is the identity automorphism.

Proof. For every \( n \geq 1 \) let \( \theta^n: \mathcal{T} \rightarrow \mathcal{T} \) be defined in the obvious way. Let \( a \)}
be an arbitrary fixed element of $L$. For $n > 1$ put $a_n = a \land \theta(a) \land \theta^2(a) \land \cdots \land \theta^n(a)$. Then $a_n \land \theta(a_n) = a_{n+1}$ ($n > 1$). Using (I) and the fact that $L$ is distributive the statement

$$\theta(x) = (x \land \theta(x)) \lor \left[ \theta(x \land \theta(x)) \right]$$

holds identically in $L$. Using this identity it is easily shown that $\theta(a) = a_1 \lor \theta(a_n)$ ($n > 1$). Thus

$$\theta(a) = \lor \{a_1 \lor \theta(a_n): n > 1\} = a_1 \lor (\lor\{\theta(a_n): n > 1\}).$$

If $c = \lor\{a_n: n > 1\}$ then

$$\theta(c) = \lor\{\theta(a_n): n > 1\} \lor \{a_{n+1}: n > 1\} \lor c$$

and by (II), $\theta(c) = c$. Hence $\theta(a) = a_1 \lor \theta(c) = a_1 \lor c = a_1 < a$ and again by (II), $\theta(a) = a$. This completes the proof.

Let $T$ be an operator on $H$. Notice that if $S$ is an invertible operator commuting with $T$ then $SM \in \Lat T$ ($M \in \Lat T$) and the mapping $M \rightarrow SM$ is a lattice automorphism with the mapping $M \rightarrow S^{-1}M$ as its inverse. If the operator $A$ commutes with $T$ and $\mu$ is a scalar with $|\mu| > ||A||$, the operator $S = A - \mu$ is invertible, commutes with $T$ and $Lat A = Lat S$. By a result of Sarason [4], $Lat S = Lat S^{-1}$. It readily follows that Hyperlat $T = Lat T$ if and only if for every invertible operator $S$ commuting with $T$ satisfying $Lat S = Lat S^{-1}$ the mapping $M \rightarrow SM$ ($M \in \Lat T$) is the identity automorphism.

**Proposition 2.2.** If $Lat T$ is distributive and $S$ is an invertible operator commuting with $T$, then $M \subseteq SM \lor S^{-1}M$ ($M \in \Lat T$).

**Proof.** Choose $\lambda$ with $0 < \lambda < 1/||S||$. The operator $C = 1 + \lambda S$ is invertible and commutes with $T$. Let $M \in \Lat T$. It is readily verified that $CM \cap SM = C(M \cap SM)$ and $CM \cap M = C(M \cap S^{-1}M)$. Since $CM \subseteq SM \lor M$, by distributivity we have

\[
CM = (CM \cap SM) \lor (CM \cap M) = C(M \cap SM) \lor C(M \cap S^{-1}M) = C(M \cap [SM \lor S^{-1}M])
\]

and the result follows.

**Theorem 2.3.** If $Lat T$ is $\sigma$-infinitely meet distributive Hyperlat $T = Lat T$.

**Proof.** By our earlier remarks it suffices to show that if $S$ is an invertible operator commuting with $T$ and satisfying $Lat S = Lat S^{-1}$ then the automorphism $M \rightarrow SM$ of $Lat T$ is the identity automorphism. Since $Lat T$ is distributive, this automorphism satisfies condition (I) of Proposition 2.1 by Proposition 2.2. Since $Lat S = Lat S^{-1}$, condition (II) is also satisfied. The result now follows from Proposition 2.1.

**Corollary 2.3.1.** Hyperlat $T = Lat T$ if $Lat T$ is any one of the following:  
(i) commutative;  
(ii) isomorphic to the direct product of complete chains;  
(iii) totally ordered.

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Proof. It is clear that in cases (ii) and (iii) Lat T is \( \sigma \)-infinitely meet distributive. Suppose that Lat T is commutative. Then it is also distributive. This follows from the fact that if \( P \) and \( Q \) are commuting projections then \( PQ \) is the projection onto the intersection of the ranges of \( P \) and \( Q \) and \( P + Q - PQ \) is the projection onto the closed linear span of the ranges of \( P \) and \( Q \). Thus if \( K, L, M \in \text{Lat } T \) then

\[
P_{K \cap (L \vee M)} = P_K P_{L \vee M} = P_K (P_L + P_M - P_L P_M) \]

\[
= P_K \cap L + P_K \cap M - P_K \cap L P_K \cap M = P_{(K \cap L) \vee (K \cap M)}.\]

Now let

\[
K, L_n (n \geq 1) \in \text{Lat } T.
\]

Then \( P \cap (L_n; n \geq 1) \), respectively \( P \cap (K \vee L_n; n \geq 1) \), is the strong limit of the sequence \( \{ P \cap (L_n; 1 \leq n \leq k); k \geq 1 \} \), respectively \( \{ P \cap (K \vee L_n; 1 \leq n \leq k); k \geq 1 \} \). But

\[
P \cap (K \vee L_n; 1 \leq n \leq k) = P_K \vee (L_n; 1 \leq n \leq k)
\]

\[
= P_K + P \cap (L_n; 1 \leq n \leq k) - P_K P \cap (L_n; 1 \leq n \leq k).
\]

Taking strong limits gives

\[
P \cap (K \vee L_n; n \geq 1) = P_K + P \cap (L_n; n \geq 1) - P_K P \cap (L_n; n \geq 1) = P_{K \vee (L_n; n \geq 1)}.\]

Hence Lat T is \( \sigma \)-infinitely meet distributive. The result follows by applying Theorem 2.3.

The results (i) and (iii) above are not new. The former was proved in [2] and the latter in [3].

3. Concluding remarks. It is a simple exercise to show for any linear transformation \( T \) on a finite-dimensional complex linear space that Hyperlat T is distributive and finite, therefore \( \sigma \)-infinitely meet distributive. It seems an interesting question whether Hyperlat T is always \( \sigma \)-infinitely meet distributive or even whether the converse of Theorem 2.3 holds. For a normal operator \( T \), Hyperlat T consists of the ranges of the spectral projections for \( T \) [2] and so is both commutative and a Boolean algebra and so is certainly \( \sigma \)-infinitely meet distributive.

References


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