ON POSITIVELY TURNING IMMERSIONS

J. R. QUINE

Abstract. Let \( \gamma : S^1 \to \mathbb{C} \) be a \( C^2 \) immersion of the circle. Let \( k \) be the number of zeros of \( \gamma \) and suppose \( d \arg \gamma(e^{i\theta})/d\theta > 0 \) for \( \gamma(e^{i\theta}) \neq 0 \); then \( \text{twn} \gamma = k/2 + (2\pi)^{-1} \int_A d\arg \gamma \) where \( \text{twn} \gamma \) is the tangent winding number, and \( A = S^1 - \gamma^{-1}(0) \). This generalizes the theorem of Cohn that if \( p \) is a self-inverse polynomial, the number of zeros of \( p' \) in \( |z| > 1 \) is the same as the number of zeros of \( p \) in \( |z| > 1 \). For \( k = 0 \), this is a topological generalization of Lucas’ theorem. We show how \( (2\pi)^{-1} \int_A d\arg \gamma \) represents a generalization of the notion of the winding number of \( \gamma \) about 0.

Introduction. The classical theorem of Lucas says that if \( p \) is a polynomial and all the zeros of \( p \) are in \( |z| < 1 \), then all the zeros of \( p' \) are in \( |z| < 1 \). We may prove the theorem topologically as follows: Since all the zeros of \( p \) are in \( |z| < 1 \), we readily see that \( d \arg p(e^{i\theta})/d\theta > 0 \) for all \( \theta \). Thus \( p(e^{i\theta}) \) is a regular closed curve such that the vector from the origin to a point on the curve always turns in a positive (counterclockwise) direction. For such a curve it is seen that the tangent winding number is equal to the winding number about zero. The tangent winding number is just \( 1 + \) (number of zeros of \( p' \) in \( |z| < 1 \)) and this proves Lucas’ theorem. In this paper, we show that a theorem of Cohn on self-inverse polynomials can be generalized in a similar way to a theorem relating the tangent winding number and the winding number about zero of a regular curve \( \gamma \) on which \( \arg \gamma \) is increasing except where \( \gamma \) is zero. The theorem that we prove has a topological generalization of Lucas’ theorem as a special case. In the course of the proof, we offer a generalization of the notion of winding number \( \omega(\gamma, a) \) to cover the case when \( a \) is on the image of \( \gamma \).

We remark that the proof given here of Cohn’s theorem is similar to a proof in Bonsall and Marden [1] that was suggested by J. L. Walsh. Other information on positively turning curves may be found in Polya and Szegö [4, part three, problems 103–111].

1. A generalization of Cohn’s theorem. Let \( p \) be a polynomial of degree \( n \). Say \( p \) is self-inverse if the zeros of \( p \) are symmetric in the circle \( |z| = 1 \), i.e., \( p(z) = 0 \) iff \( p(1/z) = 0 \). Now \( p \) is self-inverse iff \( z^n p(1/z) = cp(z) \) where \( |c| = 1 \). Let \( c = e^{2i\phi} \). Then we have

\[
p(e^{i\theta})e^{i(\phi-n\theta)/2} = p(e^{i\theta})e^{-i(\phi-n\theta)/2}
\]

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so that we may write \( p(e^{i\theta}) = R(\theta)e^{-i(\phi - n\theta/2)} \) where \( R(\theta) \) is real. Therefore for \( p(e^{i\theta}) \neq 0, \)
\[
\frac{d\arg p(e^{i\theta})}{d\theta} = \text{Im} \left( ie^{i\theta} p'(e^{i\theta}) / p(e^{i\theta}) \right) = n/2.
\]

Thus these polynomials have the interesting property that the ray from 0 to \( p(e^{i\theta}) \) turns at a constant positive angular velocity except where \( p(e^{i\theta}) = 0. \)

We note also that \( p'(z) \neq 0 \) except at the multiple zeros of \( p. \) We have the following theorem of Cohn on self-inversive polynomials (see Cohn [2] and Marden [3]): If \( p \) is self-inversive then the number of zeros of \( p' \) in \( |z| > 1 \) is equal to the number of zeros of \( p \) in \( |z| > 1. \) In what follows we will prove a generalization of Cohn's theorem to closed curves in \( \mathbb{C} \) with positively changing argument.

Let \( S^1 = \{ z | |z| = 1 \} \subseteq \mathbb{C}, \) considered as a 1-manifold with coordinate systems \( \theta \rightarrow e^{i\theta}. \) We will consider a closed curve as a \( C^2 \) map \( \gamma: S^1 \rightarrow \mathbb{C}. \) We will suppose also that \( d\gamma(z)/d\theta \neq 0 \) for \( z \in S^1, \) so that \( \gamma \) is an immersion. Let \([\gamma]\) denote the image of \( \gamma. \) If \( 0 \notin [\gamma] \) then the winding number of \( \gamma \) about 0, \( \omega(\gamma,0), \) is defined to be \( (2\pi)^{-1} \int_{S^1} d\arg \gamma. \) We may drop the condition \( 0 \notin [\gamma] \) and extend the definition, as the following discussion shows. Suppose \( \gamma(e^{i\phi}) = 0. \) Then
\[
\lim_{\theta \rightarrow \phi} \frac{d\arg \gamma(e^{i\theta})}{d\theta} = \lim_{\theta \rightarrow \phi} \text{Im} \left( \frac{d\gamma(e^{i\theta})/d\theta}{\gamma(e^{i\theta})} \right) = \lim_{\theta \rightarrow \phi} \text{Im} \left( \frac{d\gamma(e^{i\theta})/d\theta}{\gamma(e^{i\theta})} - \frac{1}{(\theta - \phi)} \right) = \text{Im} \left( \frac{d^2 \gamma(e^{i\phi})/d\theta^2}{2d\gamma(e^{i\phi})/d\theta} \right).
\]

Thus we may extend\[
\text{Im} \left( \frac{d\gamma(e^{i\theta})/d\theta}{\gamma(e^{i\theta})} \right) d\theta = d\arg \gamma(e^{i\theta})
\]
to a continuous differential on \( S^1 \) and define the integral over \( S^1 \) as \( \omega(\gamma,0). \)

We remark here for future reference that
\[
\lim_{\theta \rightarrow \phi^+} \text{Re} \left( \frac{d\gamma(e^{i\theta})/d\theta}{\gamma(e^{i\theta})} \right) = \lim_{\theta \rightarrow \phi^-} \frac{1}{\theta - \phi} \text{Re} \left( \frac{d\gamma(e^{i\theta})}{d\theta} \frac{(\theta - \phi)}{\gamma(e^{i\theta})} \right) = +\infty
\]
and similarly as \( \theta \rightarrow \phi^- \), the limit is \( -\infty. \)

Now let \( \beta = d\gamma/d\theta. \) We define the tangent winding number of \( \gamma, \) \( \text{tnw} \gamma, \) to be \( \omega(\beta,0). \) We have the following theorem.

**Theorem 1.** Suppose \( \gamma: S^1 \rightarrow \mathbb{C} \) is a \( C^2 \) immersion. Let \( k \) be the number of zeros of \( \gamma, \) and suppose \( d\arg \gamma(z)/d\theta > 0 \) for \( \gamma(z) \neq 0. \) Then \( \text{tnw} \gamma = k/2 + \omega(\gamma,0). \)

**Proof.** First suppose \( \gamma^{-1}(0) \) is not the null set, and let \( \gamma^{-1}(0) = \{ z_1, \ldots, \} \)
Let $A_j$ be the arc of $S^1$ from $z_j$ to $z_{j+1}$ for $j = 1, \ldots, k - 1$ and $A_k$ be the open arc from $z_k$ to $z_1$. Let $\beta = d\gamma/d\theta$; then $d\arg \gamma/d\theta = \text{Im} (\beta/\gamma)$ on $S^1 - \gamma^{-1}(0)$. Let $z_j = \exp(i\theta_j)$ for $j = 1, \ldots, k$. By our previous remark,

$$\lim_{\theta \to \theta_j^-} \text{Re} \left( \frac{\beta(e^{i\theta})}{\gamma(e^{i\theta})} \right) = +\infty$$

and as $\theta \to \theta_j^+$, the limit is $-\infty$. Since by hypothesis, $\text{Im} (\beta/\gamma) > 0$ on $A_j, j = 1, \ldots, k$, we conclude that $\int_{A_j} d\arg (\beta/\gamma) = \pi, j = 1, \ldots, k$. We also have $d\arg \beta = d\arg (\beta/\gamma) + d\arg \gamma$. Integrating over $S^1 - \gamma^{-1}(0)$ and dividing by $2\pi$, we get the result.

Now if $\gamma^{-1}(0)$ is empty, then since $\text{Im} (\beta/\gamma) > 0$ on $S^1$, we have $\int_{S^1} d\arg (\beta/\gamma) = 0$ and $\text{tw}(\beta) = \omega(\gamma, 0)$ and the theorem holds with $k = 0$. This completes the proof of the theorem.

We now show how Theorem 1 proves the theorem of Cohn. We remark first that it is enough to prove the theorem when none of the zeros of $p$ on $|z| = 1$ are multiple because there are self-inversive polynomials $q_e$ near $p$ with distinct roots on $|z| = 1$, and such that $\lim_{e \to 0} q_e = p$. Now $q_e$ has no zeros on $|z| = 1$ so the number of zeros of $q_e$ and $q'_e$ in $|z| > 1$ remains the same in taking the limit. Now suppose $p$ is self-inversive of degree $n$, where none of the zeros of $p$ on $|z| = 1$ are multiple. i.e., $p(z) \neq 0$ on $|z| = 1$. Let $l$ be the number of zeros of $p'$ in $|z| > 1$ and let $m$ be the number of zeros of $p$ in $|z| > 1$. Then the number of zeros of $p$ on $|z| = 1$ is $n - 2m$ and the number of zeros of $p$ in $|z| < 1$ is $n - k - 1$. Since $d\arg p(e^{i\theta})/d\theta = n/2$, the theorem and the argument principle give $n - l = (n - 2m)/2 + n/2$; hence $l = m$.

The theorem for $k = 0$ is a generalization of Lucas' theorem for polynomials.

2. The generalized winding number. We will investigate further the meaning of the generalized winding number defined in §1. We first give some definitions. Let $I = [0, 1]$ and let $I \times S^1$ have coordinate systems given by $(t, \phi) \mapsto (t, e^{i\phi})$. Let $F: I \times S^1 \to \mathbb{C}$ be $C^2$. Say $F$ is a positive monotopy if the Jacobian is positive on $I \times S^1$ in these coordinate systems.

Now let $\gamma$ be a $C^2$ immersion and suppose $F$ is a positive monotopy such that $F(t, z) = \gamma(z)$ for $z \in S^1$ and such that the only zeros of $F$ are on $\frac{1}{2} \times S^1$. Let $F(0, z) = a(z)$ and $F(1, z) = \beta(z)$. Then we define $\omega^+(\gamma, 0) = \omega(\beta, 0)$ and $\omega^-(\alpha, 0) = (\alpha, 0)$. Such a monotopy may be constructed by setting

$$F(t, z) = \gamma(z) + (t - \frac{1}{2})ci \frac{d\gamma}{d\theta}(z)$$

for suitably small $c > 0$. To show that $\omega^+$ and $\omega^-$ are defined independently of $F$ and to show how they relate to the generalized winding number we prove the next theorem.

**Theorem 2.** Let $\gamma: S^1 \to \mathbb{C}$ be a $C^2$ immersion and let $k$ be the number of zeros of $\gamma$. Then $\omega(\gamma, 0) = (\omega^+(\gamma, 0) + \omega^-(\gamma, 0))/2$; $k = \omega^+(\gamma, 0) - \omega^-(\gamma, 0)$.

**Proof.** Write $f(t, x) = F(t, e^{2\pi itx})$. Let $X = I \times I$. Then $f: X \to \mathbb{C}$. Let $X^+ = \{(t, x) | t \geq \frac{1}{2}, x \in \mathbb{R} \}$ and $X^- = \{(t, x) | t < \frac{1}{2}, x \in \mathbb{R} \}$. Let $(P_0, \ldots, P_k) = F^{-1}(0)$. By as-
sumption $F^{-1}(0) \in \frac{1}{2} \times I$. Let

$$D_j = \{ P \in I \times I | \text{dist} (P, P_k) \leq \epsilon \}, \quad j = 1, \ldots, k,$$

where $\epsilon$ is chosen so that $D_j \subseteq X$ for $j = 1, \ldots, k$. Let $\partial D_j$ be the positively oriented boundary of $D_j$. Since $f$ has positive Jacobian, we may choose $\epsilon$ so that $\int_{\partial D_j} df/f = 2\pi$, $j = 1, \ldots, k$. Let $\partial D_j^+ = \partial D_j \cap X^+$ and $\partial D_j^- = \partial D_j \cap X^-$, both considered as paths for $j = 1, \ldots, k$. Let $\partial D_j^+$ by the cycle $\sum_{j=1}^n \partial D_j^+$ and $\partial D_j^-$ be the cycle $\sum_{j=1}^n \partial D_j^-$. Let $\sigma_t$ be the cycle consisting of the line $t = \frac{1}{2}$ minus $\bigcup_{j=1}^n D_j$, traversed in the direction of increasing $x$. Since $df/f$ is a closed differential in $X - \{ P_1, \ldots, P_k \}$ we have

\begin{equation}
\omega(\alpha, 0) = \frac{1}{2\pi} \int_{\sigma_t} \frac{df}{f} + \frac{1}{2\pi} \int_{\partial D_j^-} \frac{df}{f}
\end{equation}

and

\begin{equation}
\omega(\beta, 0) = \frac{1}{2\pi} \int_{\sigma_t} \frac{df}{f} - \frac{1}{2\pi} \int_{\partial D_j^+} \frac{df}{f}.
\end{equation}

We also verify that

$$\lim_{\epsilon \to 0} \int_{\partial D_j^+} \frac{df}{f} = \pi; \quad \lim_{\epsilon \to 0} \int_{\partial D_j^-} \frac{df}{f} = \pi.$$

Now adding and subtracting (1) and (2) and taking the limit gives the theorem.

Thus if we "expand" or "contract" $\gamma$ slightly and take the average winding number about 0, we get $\omega(\gamma, 0)$.

We remark that the conclusion to Theorem 1 may now be stated as twn $\gamma = \omega^+(\gamma, 0)$.

**References**


Department of Mathematics, Florida State University, Tallahassee, Florida 32306