PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 61, Number 1, November 1976

LARGEST NORMAL NEIGHBORHOODS

V. OZOLS

ABSTRACT. It is well known that the largest normal neighborhood of a point in a compact Riemannian manifold is a Euclidean cell, that is, homeomorphic to the open unit ball. In this paper it is proved that this normal neighborhood is in fact C^∞ diffeomorphic to the open unit ball. The method is to paste together a sequence of C^∞ radial dilations which combine to engulf an open ball or all of $\mathbf{R}^n$.

Let M be a compact C^∞ Riemannian manifold of dimension n. For each $p \in M$ let $\tilde{C}(p)$ be the cut locus of p in the tangent space T_pM. Then $\tilde{C}(p)$ is homeomorphic to the $(n-1)$-sphere S^{n-1}, and the bounded component $\tilde{E}_p$ of $T_pM \backslash \tilde{C}(p)$ is homeomorphic to an open n-ball B^n in $\mathbf{R}^n$ (see [1]). It is not true in general (even when $n = 2$) that $\tilde{C}(p)$ is a smooth manifold, in which case it cannot be diffeomorphic to S^{n-1}. It is natural to ask, therefore, whether $\tilde{E}_p$ is diffeomorphic to B^n. When $n \neq 4$, it is known [2], [3], [4] that any open set in $\mathbf{R}^n$ which is homeomorphic to B^n is C^∞ diffeomorphic to B^n. The method of proof is not elementary, however, and the result is so far not known if $n = 4$. On the other hand, the sets $\tilde{E}_p$ encountered in Riemannian geometry have the additional property that they are star-shaped about $0 \in T_pM$. This property makes them more tractable, and allows us to give an elementary proof that every $\tilde{E}_p$ is C^∞ diffeomorphic to B^n with no restriction on the dimension. In fact, we will construct a diffeomorphism (a kind of "radial engulfing") which preserves directions (i.e. preserves the rays emanating from the origin).

Let $\|\cdots\|$ be the Euclidean norm in $\mathbf{R}^n$, and for each $r > 0$ let $S_r = \{x \in \mathbf{R}^n \mid \|x\| = r\}$ and $B_r = \{x \in \mathbf{R}^n \mid \|x\| < r\}$. Suppose $\mu: S_1 \to \mathbf{R}^1$ is a continuous function with $\mu(\theta) > 0$ for all $\theta \in S_1$, and define the open set $U_\mu \subset \mathbf{R}^n$ by

$$U_\mu = \{x \in \mathbf{R}^n \mid x = 0 \text{ or } 0 < \|x\| < \mu(x/\|x\|)\}.$$

LEMMA. *If $U \subset \mathbf{R}^n$ is defined by a continuous function $\mu: S_1 \to \mathbf{R}^1$ as above, then there is a C^∞ diffeomorphism $h: U \to B_1$ which preserves directions.*

PROOF. Let $\mu: S_1 \to \mathbf{R}^1$ be the defining function of U, and let $r = \inf\{\mu(\theta) |$

Received by the editors March 14, 1976.
AMS (MOS) subject classifications (1970). Primary 53C20; Secondary 57A50, 57C30.
Key words and phrases. Cut locus, normal neighborhood, engulfing.

$\theta \in S_1\}$, $R = \sup\{\mu(\theta) | \theta \in S_1\}$. Let $0 < r_0 < r$ be fixed, and choose a sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ such that: (i) $\varepsilon_i > 0$, (ii) $\varepsilon_i > \varepsilon_{i+1}$, (iii) $\varepsilon_1 < r - r_0$, (iv) $\lim_{i\to\infty} \varepsilon_i = 0$. Let $\delta_i = \varepsilon_i - \varepsilon_{i+1}$, so that $0 < \delta_i < \varepsilon_i$ and $\lim \delta_i = 0$. Define continuous functions $\tilde{\mu}_i \colon S_1 \to \mathbf{R}^1$ by $\tilde{\mu}_i = \mu - \varepsilon_i + \delta_i/2$. Thus, $\tilde{\mu}_i(\theta)$ is the midpoint between $\mu(\theta) - \varepsilon_i$ and $\mu(\theta) - \varepsilon_{i+1}$. For each i, let $\mu_i \colon S_1 \to \mathbf{R}^1$ be a C^∞ function such that $|\mu_i(\theta) - \tilde{\mu}_i(\theta)| < \delta_i/4$ for all $\theta \in S_1$. Then $\{\mu_i\}$ satisfies: (a) $r_0 < \mu_1(\theta) < \mu_2(\theta) < \cdots$; (b) $\mu_{i+1}(\theta) - \mu_i(\theta) > \delta_i/4 + \delta_{i+1}/4 = (\varepsilon_i - \varepsilon_{i+2})/4 > 0$; (c) $\mu_i \to \mu$ uniformly on S_1. For each i, let $U_i = U_{\mu_i}$ be the set defined by μ_i. Then all the sets U_i satisfy $\overline{U}_i \subset U_{i+1}$ and $U = \bigcup_{i=1}^{\infty} U_i$. Let $\mu_0 \colon S_1 \to \mathbf{R}^1$ be the constant function $\mu_0 \equiv r_0$, and $U_0 = B_{r_0}$ the set it defines. Let $r_i = \inf\{\mu_i(\theta) | \theta \in S_1\}$, $R_i = \sup\{\mu_i(\theta) | \theta \in S_1\}$. Then $r_0 < r_1 < \cdots$, and $R_0 < R_1 < \cdots$. Let $A_1 = R_1$ and define inductively $A_{i+1} = R_{i+1}(A_i/r_i)$. Then

$$A_{i+1}/R_{i+1} = A_i/r_i \geqslant A_i/R_i = A_{i-1}/r_{i-1} \geqslant \cdots \geqslant A_1/R_1 = 1.$$

Let $\alpha \colon \mathbf{R}^1 \to \mathbf{R}^1$ be a C^∞ function such that: (i) $\alpha(t) = 0$ if $t \leqslant 0$, $\alpha(t) = 1$ if $t \geqslant 1$; (ii) $\alpha'(t) > 0$ if $t \in (0,1)$. Introduce polar coordinates (θ, r) in $\mathbf{R}^n$, where $\theta \in S_1$ and $r \geqslant 0$. When $r > 0$ and θ is restricted to a coordinate patch on S_1, then (θ, r) is a coordinate patch on $\mathbf{R}^n \setminus \{0\}$. Define a sequence $\{\eta_i\}_{i=0}^{\infty}$ of maps $\eta_i \colon \mathbf{R}^n \to \mathbf{R}^1$ inductively by: $\eta_0 \equiv 1$, and

$$\eta_i(\theta, r) = \begin{cases} 1 \quad \text{if } r = 0, \\ \left(1 - \alpha\left(\dfrac{r - \mu_{i-1}(\theta)}{\mu_i(\theta) - \mu_{i-1}(\theta)}\right)\right)\eta_{i-1}(\theta, r) \\ \qquad\qquad + \alpha\left(\dfrac{r - \mu_{i-1}(\theta)}{\mu_i(\theta) - \mu_{i-1}(\theta)}\right)\dfrac{A_i}{\mu_i(\theta)} \quad \text{if } r \neq 0. \end{cases}$$

Note that for $0 < r < \mu_{i-1}(\theta)$ (i.e. $(\theta, r) \in \overline{U}_{i-1}$), we have $\eta_i(\theta, r) = \eta_{i-1}(\theta, r)$; and for $r \geqslant \mu_i(\theta)$, $\eta_i(\theta, r) = A_i/\mu_i(\theta)$. In particular, all $\eta_i = 1$ when $0 \leqslant r \leqslant r_0 = R_0$. Therefore, each $\eta_i \colon U \to \mathbf{R}^1$ is a C^∞ function. Moreover, if $(\theta, r) \in U_i \setminus U_{i-1}$, then $\eta_{i-1}(\theta, r) = A_{i-1}/\mu_{i-1}(\theta)$; and

$$A_i/\mu_i(\theta) \geqslant A_i/R_i = A_{i-1}/r_{i-1} \geqslant A_{i-1}/\mu_{i-1}(\theta).$$

Thus, along each ray $r \mapsto (\theta, r)$ (θ fixed), η_i is nondecreasing in each interval $[\mu_{i-1}(\theta), \mu_i(\theta)]$; hence nondecreasing along the entire ray emanating from 0. Moreover, $\eta_i(\theta, 0) = 1$ so $\eta_i \geqslant 1$ on U. Define maps $f_i \colon U \to \mathbf{R}^n$ by $f_i(\theta, r) = (\theta, \eta_i(\theta, r)r)$ (i.e. $f_i(x) = \eta_i(x) \cdot x$). These are C^∞ maps and $f_i | \overline{U}_{i-1} = f_{i-1} | \overline{U}_{i-1}$. Let $\eta(x) = \lim_{i\to\infty} \eta_i(x)$ and $f(x) = \lim_{i\to\infty} f_i(x)$. Since $\overline{U}_{i-1} \subset U_i$, $f | U_i = f_i | U_i$, and $\eta | U_i = \eta_i | U_i$, it follows that η and f are C^∞ maps on U. Note that the inductive definitions above make sense for all $x \in \mathbf{R}^n$, but one cannot expect η or f to be differentiable on $\partial U = \overline{U} \setminus U$ even though η_i and f_i are. Let $\theta = (\theta_1, \ldots, \theta_{n-1})$ be a coordinate chart on S_1, and compute Df in terms of the local coordinates $(\theta_1, \ldots, \theta_{n-1}, r)$ in $U \setminus \{0\}$. It is easily seen that in $U \setminus \{0\}$, we have

$$Df = \left(\begin{array}{c|c} I_{n-1} & 0 \\ \hline * & r\partial\eta_i/\partial r + \eta_i \end{array}\right)$$

where I_{n-1} is the $(n-1) \times (n-1)$ identity matrix. Thus, $\det(Df) = r\partial\eta_i/\partial r + \eta_i$. But η_i is nondecreasing in the r-direction so $\partial\eta_i/\partial r \geqslant 0$; and since $r \geqslant 0$, $\eta_i \geqslant 1$, it follows that $\det(Df) > 0$ in $U\backslash\{0\}$. In a neighborhood of 0, $f = \mathrm{id}$, so f is nonsingular on U. Since f preserves the rays $r \mapsto (\theta, r)$, and $\eta_i \geqslant 1$, it follows that f is one-to-one on U. Since $A_{i+1} = R_{i+1}A_i/r_i$ and $R_{i+1}/r_i \geqslant 1$, we have $A_1 \leqslant A_2 \leqslant \cdots$. Thus, either $\lim A_i = +\infty$ or $\lim A_i = A < +\infty$. For each $\theta \in S_1$,

$$\begin{aligned} \lim_{t \to \mu_i(\theta)^-} f(\theta, t) &= \lim f_i(\theta, t) = \lim(\theta, \eta_i(\theta, t)t) \\ &= (\theta, A_i\mu_i(\theta)/\mu_i(\theta)) = (\theta, A_i) \end{aligned}$$

(the limit is taken through values $t < \mu_i(\theta)$). Consequently, $f(U_i) = B_{A_i}$ for each $i = 1, 2, \ldots$. Therefore, either $f(U) = \mathbf{R}^n$ or $f(U) = B_A$, $A < +\infty$. It is easy to construct direction preserving C^∞ diffeomorphisms $\mathbf{R}^n \to B_1$, $B_A \to B_1$, so the proof is complete. Q.E.D.

In the case of a compact Riemannian manifold, the sets $\tilde{E}_p$ are all defined by positive continuous functions $\mu\colon S_1 \to \mathbf{R}^1$, so by the lemma:

THEOREM. *Let M be a compact C^∞ Riemannian manifold, $p \in M$ any point, $\tilde{C}(p)$ the cut locus of p in T_pM, and $\tilde{E}_p$ the bounded component of $T_pM\backslash\tilde{C}(p)$. Then there is a direction preserving C^∞ diffeomorphism $\tilde{E}_p \to B_1$.*

Since $\exp_p \tilde{E}_p$ is the largest normal neighborhood of p in M, this theorem shows that $\exp_p \tilde{E}_p$ is diffeomorphic to the open unit ball.

REFERENCES

1. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol. 2, Interscience, New York, 1969. MR **38** #6501.

2. E. E. Moise, *Affine structures in 3-manifolds*. V. *The triangulation theorem and Hauptvermutung*, Ann. of Math. (2) **56** (1952), 96–114. MR **14**, 72.

3. J. Munkres, *Obstructions to the smoothing of piecewise-differentiable homeomorphisms*, Ann. of Math. (2) **72** (1960), 521–554. MR **22** #12534.

4. J. R. Stallings, *The piecewise-linear structure of Euclidean space*, Proc. Cambridge Philos. Soc. **58** (1962), 481–488. MR **26** #6945.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98195