PERIODIC POINTS OF CONTINUOUS MAPS
AND LINDEMANN'S INDEPENDENCE THEOREM
FOR EXPONENTS

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Abstract. We give a simple proof of the sufficiency of known conditions
for the existence of periodic points of a continuous map, using a classical
theorem of Lindemann on transcendental numbers.

The purpose of this note is to give a very simple proof of a generalization
of the following well-known theorem of F. B. Fuller:

Theorem 1 (Fuller). If \( \phi \) is a self-homeomorphism of a finite complex \( X \), and
\( \chi(X) \neq 0 \), then \( \phi \) has a periodic point.

We consider a space \( X \) for which the Lefschetz Fixed Point Theorem is
valid, and a map \( \phi \colon X \to X \). Let \( \phi_\ast q \colon H_q(X) \to H_q(X) \) \((q \geq 0)\) denote the
induced maps of the rational homology of \( X \); set \( \lambda_0 = 0 \), and let \( \lambda_1, \ldots, \lambda_n \)
be the distinct nonzero eigenvalues of the \( \phi_\ast q \); let \( m_q(\lambda_j) \) be the multiplicity
of \( \lambda_j \) in \( \phi_\ast q \) \((j = 0, \ldots, n)\). We then have:

Theorem 2. If \( \phi \) is a self-map of a Lefschetz space \( X \) and \( \phi \) has no periodic
points, then

\[
\sum_q (-1)^q m_q(\lambda_0) = \chi(X) \quad \text{and} \quad \sum_q (-1)^q m_q(\lambda_j) = 0 \quad (j = 1, \ldots, n).
\]

Theorem 1 has been considerably extended by Fuller [2] and others: the
survey [1] of Fadell contains an elegant exposition of these results. Theorem 2
also follows from these extensions, but our direct proof is somewhat simpler.

Proof of Theorem 2. Let \( m \) be the maximum dimension for which
\( H_q(X) \neq 0 \). For \( q = 0, \ldots, m \), let \( A_q \) be a matrix representing \( \phi_\ast q \) with
respect to some fixed basis for \( H_q(X) \). Define

\[
B_k = \text{diag}[A_0^k, -A_1^k, \ldots, (-1)^m A_m^k],
\]

for \( k \geq 0 \), and set

\[
E = \sum_{k=0}^{\infty} \frac{1}{k!} B_k.
\]

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Note that $\text{Tr}(B_0) = \chi(X)$, and, for $k \geq 1$, $\text{Tr}(B_k) = \Lambda(\phi^k)$, the Lefschetz number of $\phi^k$. If $\phi$ has no periodic points then, by the Lefschetz Fixed Point Theorem, $\text{Tr}(B_k) = 0$ for $k \geq 1$. We then have, equating traces in (1):

$$
(2) \quad \sum_{j=0}^{n} \sum_{q=0}^{m} (-1)^q m_q(\lambda_j)e^{\lambda_j} = \chi(X) \cdot e^{\lambda_0}.
$$

But the $A_q$ are rational matrices, so the eigenvalues $\lambda_0, \ldots, \lambda_n$ are distinct algebraic numbers, and we can apply the theorem of Lindemann ([3, p. 117], e.g.) that asserts the linear independence over the field of algebraic numbers of $e^{\lambda_0}, e^{\lambda_1}, \ldots, e^{\lambda_n}$, to conclude that each of the coefficients in equation (2) vanishes. This is just the conclusion of Theorem 2.

Bibliography


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