

A MEAN ERGODIC THEOREM FOR FAMILIES OF CONTRACTIONS IN HILBERT SPACE

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ABSTRACT. Let G be an LCA group and h a Hilbert space, and let $T(g)$ be a function on G into the contractions on H . Let $\{\sigma_n\}$ be a sequence of probability measures on G . Under suitable conditions on $T(g)$ and the sequence $\{\sigma_n\}$ we prove the strong convergence of the sequence $T_n = \int T(g)\sigma_n(dg)$. In certain cases we identify the limiting operator.

Let G be an LCA group and $\mathfrak{B}(G)$ the Borel sets of G . \mathcal{Q} will denote the class of functions $l(\gamma)$ defined on \hat{G} which are Fourier transforms of a probability measure defined on $\mathfrak{B}(G)$. If $\{\sigma_n\}_{n=1}^\infty$ is a sequence of such probability measures, with $\lim_n \sigma_n(\gamma) = l(\gamma)$ for $\forall \gamma \in \hat{G}$, we shall call the sequence $l(\gamma)$ -*amenable*.

A set $S \in \mathfrak{B}(G)$ is called *amenable* if \exists a nonnegative Borel measure μ on $\mathfrak{B}(G)$ and a sequence $\{S_n\}$ of Borel subsets of S such that

- (i) $\mu(B + g) = \mu(B)$ for every $B \in \mathfrak{B}(G)$ and $g \in S$,
- (ii) $0 < \mu(S_n) < n = 1, 2, \dots$,
- (iii) $\lim_{n \rightarrow \infty} \mu[(S_n + g) \cap S_n] / \mu(S_n) = 1$ for $\forall g \in S$.

If

$$l_S(\gamma) = \begin{cases} 1 & \text{if } \langle g, \gamma \rangle = 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } \forall g \in S,$$

then it is clear that $l_S \in \mathcal{Q}$ and, in fact, if $\sigma_n(\cdot) = \mu[S_n \cap (\cdot)] / \mu[S_n]$, then $\lim_n \hat{\sigma}_n(\gamma) = l_S(\gamma)$ for $\forall \gamma \in \hat{G}$.

Now let $T(g)$ be a function defined on G whose values are contractions on a Hilbert space. We shall assume that $T(g)$ is positive definite. For a definition of a positive definite valued function, see Foiaş and Nagy [2, p. 25].

With these definitions we can state our main result.

THEOREM . *Let G be an LCA group and H a Hilbert space. Let $T(g)$ be a positive definite function from G to the bounded operators on H such that $T(e)$ is the identity operator and $T(g)$ is weakly continuous. Then*

- (i) $T(g)$ is strongly continuous.
- (ii) For a probability measure σ on G the integral $\int_G T(g)\sigma(dg)$ exists strongly and is a contraction operator on H .

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(iii) With every function $l(\gamma) \in \mathcal{Q}$ is associated a contraction operator L^l on H so that if $\{\sigma_n\}_{n=1}^\infty$ is an $l(\gamma)$ -sequence of probability measures on G , then $L^l = \text{strong-lim}_{n \rightarrow \infty} \int_G T(g)\sigma_n(dg)$.

(iv) If S is an amenable set and $T(g + g') = T(g) \cdot T(g')$ for $g, g' \in S$, and if $\{\sigma_n\}_{n=1}^\infty$ is an $l_S(\gamma)$ -sequence of probability measures on G , then $\text{strong-lim}_{n \rightarrow \infty} \int_G T(g)\sigma_n(dg) = P_S$ where P_S is the orthogonal projection onto the closed linear manifold $\tilde{S} = \{h \in H | T(g)h = h \text{ for } \forall g \in S\}$.

OUTLINE OF THE PROOF. The use of the *dilation theorem* in [2, p. 25] will be indicated by (*).

(i) as defined in (*) we have

$$\| [T(g) - T(g')]h \|_H^2 = ([U(g) - U(g')]h, [T(g) - T(g')]h)_K.$$

Now use Schwartz's inequality and (*).

(ii) follows from (i).

(iii) Let $E(\cdot)$ be the resolution of the identity corresponding to $U(g)$ (as defined in (*)). Define for $l \in \mathcal{Q}$: $N^l = \int_G l(\gamma)E(d\gamma)$; clearly N^l is a contraction on K . Now let $L^l = P_H N^l$; P_H is the orthogonal projection onto H . If $\{\sigma_n\}_{n=1}^\infty$ is an $l(\gamma)$ -sequence and $h \in H$, then

$$\begin{aligned} & \left\| \int_G T(g)h\sigma_n(dg) - P_H N^l h \right\|_H^2 \\ &= \left(\int_G U(g)h\sigma_n(dg) - N^l h, \int_G T(g)h\sigma_n(dg) - P_H N^l h \right)_K \\ &\leq \left\| \int_G U(g)h\sigma_n(dg) - N^l h \right\|_K^2 \cdot 2 \cdot \|h\|_H. \end{aligned}$$

Now use the same technique as in the proof of (*) to finish the proof.

(iv) We have to show that $L^{l^S} = P_{\tilde{S}}$ (using (iii)). Since S is amenable the sequence of probability measures $\mu_n(\cdot) = \mu(S_n \cap (\cdot))/\mu(S_n)$ is an l_S -sequence. Now let $h \in H$ and $f \in \tilde{S}$; then

$$\begin{aligned} (h - L^{l^S}h, f)_H &= \left(h - \lim_{n \rightarrow \infty} \int_G T(g)h\mu_n(dg), f \right)_H \\ &= (h, f)_H - \lim_{n \rightarrow \infty} \frac{1}{\mu(S_n)} \int_{S_n} (h, T^*(g)f)_H \mu(dg) = 0. \end{aligned}$$

Therefore $h - L^{l^S}h \in (\tilde{S})^\perp$. Now let $g' \in S$; then

$$\begin{aligned} T(g')L^{l^S}h &= T(g') \text{strong-lim}_{n \rightarrow \infty} \int_G T(g)h\mu_n(dg) \\ &= \lim_{n \rightarrow \infty} \frac{1}{\mu(S_n)} \int_{S_n} T(g')T(g)h\mu(dg). \end{aligned}$$

Therefore,

$$\begin{aligned}
& L^s h - T(g')L^s h \\
&= \lim_{n \rightarrow \infty} \left[\frac{1}{\mu(S_n)} \int_{S_n} T(g) h \mu(dg) - \frac{1}{\mu(S_n)} \int_{S_n} T(g' + g) h \mu(dg) \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{\mu(S_n)} \int_{S_n \triangle (S_n + g')} T(g) h \mu(dg).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|L^s h - T(g')L^s h\|_H &= \lim_{n \rightarrow \infty} \left\| \frac{1}{\mu(S_n)} \int_{S_n \triangle (S_n + g')} T(g) h \mu(dg) \right\|_H \\
&\leq \lim_{n \rightarrow \infty} \frac{\mu[S_n \triangle (S_n + g')]}{\mu(S_n)} \|h\|_H = 0.
\end{aligned}$$

Therefore $L^s h \in \tilde{S}$ which together with $h - L^s h \in (\tilde{S})^\perp$ implies $L^s = P_{\tilde{S}}$.
Q.E.D.

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