THE CONJUGACY PROBLEM
FOR HNN GROUPS AND THE WORD PROBLEM
FOR COMMUTATIVE SEMIGROUPS

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Abstract. We show how a positive solution to the word problem for finitely
generated commutative semigroups leads to a positive solution to the
conjugacy problem for a class of HNN groups.

Here we answer a question of Professor Wilhelm Magnus concerning the
solvability of the conjugacy problem of the groups VA discussed in [1]. These
groups are given by
(I) \( \langle a_1, \ldots, a_k, b; a_i^{-1}b\rho a_i = b^{q_i}, \ldots, a_k^{-1}b\rho a_k = b^{q_k} \rangle \)
where \( p_i q_i \geq 1 \) and \( (p_i, q_i) = 1 \). We call the groups given by (I) \( VA \)-groups
and denote them by \( G(p_1, q_1, \ldots, p_k, q_k) \) as in [1]. We call the integers
appearing in (I) the \textit{exponents} of the group. Let \( l \) and \( m \) be nonzero integers
and call \( m \) \textit{reachable} from \( l \) with respect to the exponents if there is a
sequence of integers beginning with \( l \) and ending with \( m \), such that for
successive terms \( l_i \) and \( l_{i+1} \) either \( l_{i+1} = l_i(q_j/p_j) \) or \( l_{i+1} = l_i(p_j/q_j) \). The
\textit{reachability problem} for the exponents is to decide for arbitrary such \( l \) and \( m \)
whether \( m \) is reachable from \( l \).

From [1, Lemma 1] we obtain

\begin{lemma}
A \( VA \)-group has solvable conjugacy problem if and only if the
reachability problem for its exponents is solvable.
\end{lemma}

The reachability problem for the exponents of \( VA \)-group is equivalent to
the reachability problem for a class of self-dual vector addition systems [1].
Recently several computer scientists have solved the reachability problem for
self-dual vector addition systems using combinatorial methods. Below we give
a simple self-contained algebraic proof of the solvability of the reachability
problem for the exponents. This allows us to prove

\begin{theorem}
\( VA \)-groups have solvable conjugacy problem.
\end{theorem}

It suffices to consider the reachability problem restricted to the set \( M \) of
positive integers whose prime divisors are among the prime divisors of the
exponents \( p_1, q_1, \ldots, p_k, q_k \). Observe that \( M \) forms a multiplicative semi-
group and the reachability relation induces a congruence on \( M \). Denote the

finitely generated commutative semigroup resulting from this congruence by $S(p_1, q_1, \ldots, p_k, q_k)$.

We may construct a presentation of $S(p_1, q_1, \ldots, p_k, q_k)$ in the following manner. Let $d_0, \ldots, d_n$ denote the positive prime divisors of the exponents. Let $p_i$ and $q_i$ have prime decomposition

$$p_i = d_0^{e_{i0}} \cdots d_n^{e_{in}}, \quad q_i = d_0^{e'_{i0}} \cdots d_n^{e'_{in}},$$

where $e_j, e'_j > 0$ for $j = 0, \ldots, n$. We associate a generating symbol $x_i$ with $d_i$ for $i = 0, \ldots, k$. The exponents $p_i, q_i$ are seen to be coded, respectively, as the words $C_i, D_i$ where

$$C_i = x_0^{e_{i0}} \cdots x_n^{e_{in}}, \quad D_i = x_0^{e'_{i0}} \cdots x_n^{e'_{in}}.$$

Then $S(p_1, q_1, \ldots, p_k, q_k)$ has the following presentation:

$$\langle x_0, \ldots, x_n; x_u x_v = x_v x_u, C_i = D_i, 0 \leq u, v \leq n, 1 \leq i \leq k \rangle.$$

Hence, we have proved

**Lemma 2.** The reachability problem for the exponents of $G(p_1, q_1, \ldots, p_k, q_k)$ is reducible to the word problem for the finitely presented commutative semigroup $S(p_1, q_1, \ldots, p_k, q_k)$.

It has been pointed out to the author by Gilbert Baumslag and George Bergman that finitely generated commutative semigroups have solvable word problem. Baumslag observes that a finitely generated commutative semigroup $S$ is embedded in the integral semigroup ring $R$ formed from $S$. $R$ is a finitely generated commutative ring and is therefore residually finite [3, pp. 64–65] from which it follows that $S$ is residually finite. By a theorem of L. Rédei [4], $S$ is finitely related and, hence, finitely presented. It is well known that finitely presented residually finite algebras have solvable word problem (a result which is attributed by some workers in the field to Verena Dyson). Hence $S$ has solvable word problem. Our Theorem is now immediate from Lemmas 1 and 2 and the preceding remark.

Generalizations to the $HNN$ groups considered in [2] are straightforward.

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**References**


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