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ALMOST-COMPLEX SUBSTRUCTURES ON THE SPHERE

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ABSTRACT. The paper solves completely the existence problem of almost-complex substructures on spheres.

1. **Introduction**. An almost-complex s-substructure on an orientable n-manifold M is defined to be a $2s$-dimensional sub-bundle of the tangent-bundle $T(M)$ that admits an almost-complex structure, i.e. a (continuous) structure map J, $J^2 = -1$. Assuming a metric on $T(M)$, we call an almost-complex substructure J normalized iff J is also orthogonal on the sub-bundle of its definition. If J is an almost-complex s-substructure, then the orthogonal-component in its polar-decomposition (into an orthogonal and a positive-definite symmetric transformation) defines a normalized almost-complex s-substructure. Analogously, if w is a 2-form on M of constant rank $2s$, the union of the $2s$-planes on which w is of maximal rank forms a sub-bundle S_w of $T(M)$, and the orthogonal-component of w (regarded as a nonsingular, skew-adjoint transformation on S_w) defines a normalized almost-complex s-substructure (e.g. refer to [3]). We thus have: Existence of an almost-complex s-substructure $\Leftrightarrow$ Existence of a normalized almost-complex s-substructure $\Leftrightarrow$ Existence of a 2-form on M of constant rank $2s$.

The purpose of this paper is to solve completely the existence problem of almost-complex substructures on spheres. In particular, we shall prove the following Theorem.

Let c_k = dimension of the irreducible (ungraded) representation module for the Clifford-Algebra C_k, i.e. $c_k = 2^{e_k}$ where e_k = number of integers t, $0 < t \leqslant k$, $t = 0, 1, 2, 4 \pmod 8$.

Let M_k be the Atiyah-Todd number defined in [2], i.e. if for an integer n and a prime p, $v_p(n)$ denotes the exponent of p in the prime-factorization of n, then

$$v_p(M_k) = \begin{cases} \sup(r + v_p(r)), \quad 1 \leqslant r \leqslant [(k-1)/(p-1)] & \text{if } p \leqslant k, \\ 0 & \text{if } p > k. \end{cases}$$

Let L_k be the 2-component of M_k, i.e.

$$L_k = 2^{v_2(M_k)} \quad \text{where } v_2(M_k) = \sup(r + v_2(r)),\ 1 \leqslant r \leqslant k-1.$$

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THEOREM. I. *The only almost-complex substructures on even-spheres are the almost-complex structures on* S^2 *and* S^6.

II. (i) *For* $s < (2n-1)/4$; S^{2n-1} *admits an almost-complex s-substructure iff* $c_{2s}/2n$. (ii) *For* $s > (2n-1)/4$; S^{2n-1} *admits an almost-complex s-substructure iff* L_{n-s}/n.

I, II(i) and II(ii) for n odd and $n = 2, 4$ are either known, or else, easy consequences of known material and will be quickly dealt with in §2. The main task of the paper will be to prove II(ii) for n even, $n > 4$.

§3 will reduce the problem to one which can be effectively tackled; and §4 will further reduce it to the order of the obstruction $w_{n,n-s}$ of the complex Stiefel-Fibering,

$$W_{n,n-s} \xrightarrow{W_{n-1,n-s-1}} S^{2n-1}.$$

The Theorem will then follow from the case $p = 2$ of Theorem I of [4].

2. **Elimination of certain special cases**. The main tools for this section are the following:

LEMMA 2.1. *If* $2k \leqslant n$, *then* S^n *admits a field of k-planes iff it admits a field of k-frames.*

PROOF. Refer to [5, p. 144, Theorem 27.16].

COROLLARY. *Let* S^n *admit an s-substructure.*

(i) *If* $s \leqslant n/4$, *it admits a field of* $2s$*-frames.*(ii) *If* $s > n/4$, *it admits a field of* $(n-2s)$*-frames.*

LEMMA 2.2. S^{2n-1} *admits an* $(n-1)$*-substructure.*

PROOF. The canonical complex-structure i on R^{2n} defines a 1-frame, $x \mapsto ix$, $x \in S^{2n-1}$, on S^{2n-1} whose orthogonal-complement in $T(S^{2n-1})$ is preserved under i and thus defines an $(n-1)$-substructure on S^{2n-1}.

We now proceed to prove the easy part of the Theorem.

PROOF OF I. It is well known that S^2 and S^6 are the only even dimensional spheres that admit almost-complex structures. If S^{2n} admits a proper substructure (i.e. $s < n$), then by the Corollary to Lemma 2.1 it admits at least a 1-frame, which is impossible.

PROOF OF II(i). It is a consequence of the Corollary to Lemma 2.1 and the result of Adams (e.g. refer to [1]) that S^{2n-1} admits a field of 2s-frames iff $c_{2s}/2n$.

PROOF OF II(ii) FOR n ODD. Suppose S^{2n-1} admits an s-substructure for n odd and $s > (2n-1)/4$. By the Corollary to Lemma 2.1, it admits a field of $(2n-2s-1)$-frames; and $2n-2s-1 \leqslant 1$, i.e. $s \geqslant n-1$ by [1]. Applying Lemma 2.2 yields $s = n-1$. Since $L_1 = 1$, $L_2 = 2$, this is, precisely, what the Theorem states.

PROOF OF II(ii) FOR $n = 2, 4$. Obvious, since S^{2n-1} is parallelisable.

3. **Reduction of the problem.**

3.1. *Homotopy exact sequence of* $SO(2n-1) \to SO(2n) \to^p S^{2n-1}$. Consider the exact sequence

$$\begin{array}{ccccccc} \pi_{2n-1}SO(2n) & \xrightarrow{p_\#} & \pi_{2n-1}S^{2n-1} & \to & \pi_{2n-2}SO(2n-1) & \to & \pi_{2n-2}SO(2n) \\ & & \| & & & & \| \\ & & Z & & & & \pi_{2n-2}SO \\ & & & & & & \| \\ & & \text{gen. } \iota_{2n-1} & & & & 0 \; n \text{ even} \end{array}$$

By Theorem 23.4 of [5], $2\iota_{2n-1} \in \operatorname{Im} p_\#$. Hence $\pi_{2n-2}SO(2n-1) = 0$ or Z_2 generated by $\partial(\iota_{2n-1})$. $\pi_{2n-1}SO(2n-1) = 0$ iff p admits a cross-section iff S^{2n-1} is parallelisable. Thus, for n even, $n > 4$,

$$\pi_{2n-2}SO(2n-1) = Z_2.$$

As remarked in the introduction, S^{2n-1} admits an s-substructure iff it admits a normalized s-substructure. A normalized s-substructure on S^{2n-1} is, simply, a reduction of the structure-group of $T(S^{2n-1})$ from $SO(2n-1)$ to $U(s) \times SO(2n-2s-1)$; and is, thus, a cross-section to the fibration:

(i) $$SO(2n)/U(s) \times SO(2n-2s-1) \xrightarrow{SO(2n-1)/U(s)\times SO(2n-2s-1)} S^{2n-1}.$$

Define $\Gamma_{m,s} = SO(m)/U(s)$ for $2s \leqslant m$. We have a fibration:

(ii) $$\Gamma_{2n,s} = SO(2n)/U(s) \xrightarrow{\Gamma_{2n-1,s} = SO(2n-1)/U(s)} S^{2n-1}.$$

Let $V_{n,k} = SO(n)/SO(n-k)$ be the Stiefel-Manifold of orthonormal k-frames in R^n. We have a third fibration:

(iii) $$V_{2n,2s+1} \xrightarrow{V_{2n-1,2s}} S^{2n-1}.$$

The following proposition will reduce the problem to one which can be effectively tackled.

PROPOSITION 3.2. *For n even, $n > 4$, $s > (2n-1)/4$, there exists an s-substructure on S^{2n-1} iff the fibration* (ii) *admits a cross-section.*

PROOF. Let

$$j\colon U(s) \subset SO(2n-1) \quad \text{and} \quad k\colon SO(2n-2s-1) \subset SO(2n-1)$$

be the inclusions. Let

$$\pi_{2n-2}U(s) \oplus \pi_{2n-2}SO(2n-2s-1) \xrightarrow{(j_\#, k_\#)} \pi_{2n-2}SO(2n-1)$$

$$\xrightarrow{\gamma_\#} \pi_{2n-2}SO(2n-1)/U(s) \times SO(2n-2s-1),$$

$$\pi_{2n-2}U(s) \xrightarrow{j_\#} \pi_{2n-2}SO(2n-1) \xrightarrow{\beta_\#} \pi_{2n-2}\Gamma_{2n-1,s},$$

$$\pi_{2n-2}SO(2n-2s-1) \xrightarrow{k_\#} \pi_{2n-2}SO(2n-1) \xrightarrow{v_\#} \pi_{2n-2}V_{2n-1,2s}$$

be the exact homotopy sequences of the projections:

$$\gamma\colon SO(2n-1) \to SO(2n-1)/U(s) \times SO(2n-2s-1),$$
$$\beta\colon SO(2n-1) \to \Gamma_{2n-1,s} \quad \text{and} \quad v\colon SO(2n-1) \to V_{2n-1,2s}.$$

Let $a_n \in \pi_{2n-2}SO(2n-1) = Z_2$ be the generator and also the obstruction to cross-sectioning

$$p\colon SO(2n) \xrightarrow{SO(2n-1)} S^{2n-1}.$$

Then the obstructions to cross-sectioning (i), (ii), and (iii) are $\gamma_\#(a_n)$, $\beta_\#(a_n)$ and $v_\#(a_n)$, respectively. The work of Adams [1] shows that S^{2n-1} does not admit a field of $2s$-frames for $n > 4$ and $s > (2n-1)/4$. Thus, (iii) cannot be sectioned and hence $v_\#(a_n) \neq 0$. Since $\pi_{2n-2}SO(2n-1) = Z_2$, $v_\#$ is a monomorphism. Thus, $k_\# = 0$ by exactness. Hence (i) admits a cross-section iff $\gamma_\#(a_n) = 0$, iff $a_n \in \text{Image}(j_\#, k_\#)$, iff $a_n \in \text{Image}\, j_\#$, iff $\beta_\#(a_n) = 0$, iff (ii) admits a cross-section. Q.E.D.

Cross-sections to (ii) are normalized s-substructures on S^{2n-1} whose subbundles of definition have "trivial" orthogonal-complements in $T(S^{2n-1})$. Such s-substructures will be called complement-split. We now slightly generalize Proposition 3.2.

PROPOSITION 3.3. *For $s > (2n-1)/4$, S^{2n-1} admits an s-substructure iff it admits a complement-split s-substructure.*

PROOF. For n odd, the only possible $s > (2n-1)/4$ such that S^{2n-1} admits an s-substructure is $s = n-1$; and a complement-split $(n-1)$-substructure is constructed on S^{2n-1} in Lemma 2.2. The proof for n even, $n > 4$ is furnished by Proposition 3.2, and is obvious for $n = 2, 4$. Q.E.D.

4. **The boundary-operator**. Since $U(n) \cdot SO(2n-1) = SO(2n)$ and $U(n) \cap SO(2n-1) = U(n-1)$,

$$\Gamma_n = SO(2n)/U(n) = SO(2n-1)/U(n-1).$$

Let

$$u\colon SO(2n-1) \xrightarrow{U(n-1)} \Gamma_n \to \pi_{2n-1}SO(2n-1) \xrightarrow{u_\#} \pi_{2n-1}\Gamma_n$$
$$\xrightarrow{\partial_u} \pi_{2n-2}U(n-1) \to \dots$$

be the exact homotopy sequence of u. We have two more fibrations:

(iv) $$W_{n,n-s} = U(n)/U(s) \xrightarrow{W_{n-1,n-s-1} = U(n-1)/U(s)} S^{2n-1}.$$

(v) $$\Gamma_{2n-1,s} = SO(2n-1)/U(s) \xrightarrow{W_{n-1,n-s-1} = U(n-1)/U(s)} \Gamma_n$$
$$= SO(2n-1)/U(n-1).$$

Let $w_{n,n-s} \in \pi_{2n-2}W_{n-1,n-s-1}$ denote the obstruction to cross-sectioning fibration (iv). The purpose of this section is to determine the image of ∂_u, and hence further reduce the problem to an equivalent statement about the order of $w_{n,n-s}$.

Let H, K be closed subgroups of a compact topological group G with $HK = KH$, so that HK is a compact subgroup of G. Let

$$p^1\colon HK/H \cap K \xrightarrow{K/H\cap K} HK/K = H/H \cap K,$$

$$p^2\colon HK/H \cap K \xrightarrow{H/H\cap K} HK/H = KH/H = K/H \cap K$$

be the projections. Let ∂, ∂^H, ∂^K be the boundary-operators of the fibrations:

$$t\colon HK \xrightarrow{H\cap K} HK/H \cap K, \qquad t^1\colon H \xrightarrow{H\cap K} H/H \cap K, \qquad t^2\colon K \xrightarrow{H\cap K} K/H \cap K$$

respectively. We state the following

PROPOSITION 4.1. (i) $\pi_q HK/H \cap K = \pi_q H/H \cap K \oplus \pi_q K/H \cap K$ *where* $p^1_\#$ *and* $p^2_\#$ *define the projections onto the factors.* (ii) *Relative to this decomposition,* ∂ *decomposes as a sum of operators*:

$$\partial = \partial^H \oplus \partial^K.$$

PROOF. Elementary exercise.

Let $q^1\colon HK \to^K H/H \cap K$, $q^2\colon HK \to^H K/H \cap K$ denote the projections, i.e. $q^1 = p^1 t$ and $q^2 = p^2 t$.

COROLLARY 4.1. *The diagram below is ANTI-commutative.*

$$\begin{array}{ccc} \pi_q HK & \xrightarrow{q^1_\#} & \pi_q H/H \cap K \\ \downarrow q^2_\# & & \downarrow \partial^H \\ \pi_q K/H \cap K & \xrightarrow{\partial^K} & \pi_{q-1} H \cap K \end{array}$$

PROOF. Let $x \in \pi_q HK$. By Proposition 4.1 $0 = \partial t_\#(x) = \partial^H p^1_\# t_\#(x) + \partial^K p^2_\# t_\#(x) = \partial^H q^1_\#(x) + \partial^K q^2_\#(x)$ i.e. $\partial^H q^1_\# = -\partial^K q^2_\#$. Q.E.D.

PROPOSITION 4.2. *Let* $u\colon SO(2n-1) \to^{U(n-1)} \Gamma_n$ *and* $\partial_u\colon \pi_{2n-1}\Gamma_n \to \pi_{2n-2}U(n-1)$ *be the boundary-homomorphism in the homotopy exact sequence of* u. *Then* Image ∂_u *is the subgroup generated by* $2w_{n,n}$, *where* $w_{n,n}$ *is the obstruction to a cross-section of* $U(n) \to^{U(n-1)} S^{2n-1}$.

PROOF. Consider the following diagram.

$$\begin{array}{ccccc} \pi_{2n-1}SO(2n) & \xrightarrow{r_\#} & \pi_{2n-1}\Gamma_n & \xrightarrow{\partial_r} & \pi_{2n-2}U(n) = 0 \\ \downarrow p_\# & & \downarrow \partial_u & & \\ \pi_{2n-1}S^{2n-1} & \xrightarrow{\partial_q} & \pi_{2n-2}U(n-1) & & \end{array}$$

The row is part of the homotopy exact sequence of $r\colon SO(2n) \to^{U(n)} \Gamma_n$. The square is anticommutative, and is obtained from Corollary 4.1 by taking

$$H = SO(2n-1), \quad K = U(n),$$

$$G = HK = KH = SO(2n), \quad H \cap K = U(n-1).$$

By exactness, $r_\#$ is an epimorphism. Hence

$$\operatorname{Im} \partial_u = \operatorname{Im}(\partial_u r_\#) = \operatorname{Im}(\partial_q p_\#);$$

now use 3.1 and the fact that $w_{n,n} = \partial_q \iota_{2n-1}$. Q.E.D.

PROPOSITION 4.3. *For n even, $n > 4$, the fibration* (ii) *admits a cross-section iff $w_{n,n-s}$ is of odd order.*

PROOF. Consider the following commutative diagram, where the bottom row is part of the homotopy exact sequence of the fibration (v).

$$\begin{tikzcd}
\pi_{2n-1}S^{2n-1} \arrow[r, "\partial_q"] \arrow[d, "\partial_2"] \arrow[dr, "\partial_4"] & \pi_{2n-2}U(n-1) \arrow[d, "b_\#"] & \\
\pi_{2n-2}\Gamma_{2n-1,s} \arrow[r, "j_\#"] & \pi_{2n-2}W_{n-1,n-s-1} \arrow[r, "\partial_5"] & \pi_{2n-1}\Gamma_n \arrow[ul, "\partial_u"']
\end{tikzcd}$$

(ii) admits a cross-section iff $\partial_2 = 0$ iff $j_\# \partial_4 = 0$ iff $w_{n,n-s} \in \operatorname{Im} \partial_5$ and $\operatorname{Im} \partial_5 = b_\#(\operatorname{Im} \partial_u)$. By Proposition 4.2 above and the fact that $b_\#(w_{n,n}) = w_{n,n-s}$, $b_\#(\operatorname{Im} \partial_u)$ is the subgroup generated by $2w_{n,n-s}$. Hence $w_{n,n-s} \in \operatorname{Im} \partial_5$ iff it is of odd order. Q.E.D.

Finally, let p be a prime. Theorem I of [4] states that the order of the obstruction $w_{n,k}$ is prime to p iff n is divisible by the p-primary-component of M_k.

Thus, the proof of II(ii) for n even, $n > 4$, of the Main Theorem follows from Propositions 3.2 and 4.3 and the case $p = 2$ of Theorem I of [4].

The proof of the Main Theorem for $s > (2n-1)/4$ is "existential", and, unfortunately, does not give explicit constructions, except for the $(n-1)$-substructure on S^{2n-1} defined in Lemma 2.2, using the complex-structure on R^{2n}. Analogously, for n even, one can construct an $(n-2)$-substructure on S^{2n-1} using the quaternionic structure on R^{2n}; and these are the only constructions known to the author. It is hoped that one will explicitly construct s-substructures on S^{2n-1} for $s > (2n-1)/4$ whenever possible, i.e. when L_{n-s}/n; and this will throw more light onto the problem.

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