STRICT TOPOLOGY AND P-SPACES

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ABSTRACT. For a completely regular Hausdorff space $X$ and a normed space $E$, let $C_b(X, E)$ be the space of all bounded continuous functions from $X$ into $E$ with strict topology $\beta_0$. It is proved that if $X$ is a $P$-space, $(C_b(X, E), \beta_0)$ is Mackey; if, in addition, $E$ is complete, then $(C_b(X, E), \beta_0)$ is strongly Mackey.

In this paper, $X$ denotes a completely regular Hausdorff space, $K$ the field of real or complex numbers (we shall call them scalars), $C_b(X)$ all scalar-valued bounded continuous functions on $X$, $(E, \|\cdot\|)$ a normed space over $K$, $C_b(X, E)$ all bounded continuous functions from $X$ into $E$, and $E'$ the topological dual of $X$. We denote by $\langle \cdot, \cdot \rangle$ the natural bilinear form on $E \times E'$ or $E' \times E$. All vector spaces are taken over $K$. Let $\mathcal{B}(X)$ be all Borel subsets of $X$ and $M_1(X)$ all tight scalar-valued Borel measures on $X$ [1], [4], [10]. We put

$$M_1(X, E') = \{\mu: \mathcal{B}(X) \to E': \mu \text{ finitely additive},$$

and $|\mu| \in M_1(X)$, where for any $B \in \mathcal{B}(X)$, $|\mu|(B) = \sup \{\sum |\mu(B_i), x_i|: \{B_i\} \text{ a finitely Borel partition of } B \text{ and } \{x_i\} \subset E \text{ with } \|x_i\| \leq 1, \forall i\}$

(see [1], [4]). For a $\mu \in M_1(X, E')$ and $x \in E$, $\mu_x: \mathcal{B}(X) \to K$, defined by $\mu_x(B) = \langle \mu(B), x \rangle$, $B \in \mathcal{B}(X)$, is in $M_1(X)$. Integration with respect to a $\mu \in M_1(X, E')$ is taken in the sense of [1]. For a $\mu \in M_1(X, E')$ and $f \in C_b(X, E)$, $|\mu(f)| \leq |\mu| (\|f\|)$, where $\|f\|: X \to R$, $\|f\|(x) = \|f(x)\| [1, p. 851].$

The strict topology $\beta_0$ on $C_b(X, E)$ is defined by the family of seminorms $\|\cdot\|_h$, as $h$ varies through all scalar-valued functions on $X$, vanishing at infinity, $\|f\|_h = \sup_{x \in X} |h(x)f(x)|$, $f \in C_b(X, E)$. It is proved in [1] that $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_0)$, $(C_b(X, E), \beta_0)' = M_1(X, E')$, and $\beta_0$ is the finest locally convex topology which coincides with compact-open topology on norm-bounded subsets of $C_b(X, E)$; also bounded subsets of $(C_b(X, E), \beta_0)$.
are norm-bounded. (For $E = K = R$ this result is proved in [9], but it immediately carries over to the case when $E$ is a normed space since $M_t(X, E')$ is a closed subspace of the Banach space $(C_b(X, E), \|\cdot\|')$.) Considering $M_t(X, E')$ a Banach space, with norm induced by $(C_b(X, E), \|\cdot\|')$, we have $\|\mu\| = \|\mu(X)\|, \forall \mu \in M_t(X, E')$ (it is a simple verification, cf. [4, p. 315]).

A completely regular Hausdorff space $X$ is called a $P$-space if every $G_\delta$ set is open in $X$ [2, p. 63]. In this paper we prove that if $X$ is a $P$-space then $(C_b(X, E), \beta_0)$ is Mackey; if, in addition, $E$ is complete, then $E$ is strongly Mackey. A Hausdorff locally convex space $G$ is called strongly Mackey if every $\sigma(G', G)$ relatively countably compact subset of $G'$ is equicontinuous (we refer to [8] for locally convex spaces).

We first prove the following lemmas.

**Lemma 1.** Let $2^N$ denote all subsets of $N$, with product topology. If $\lambda_n : 2^N \to K$ is a sequence of countably additive measures (this implies they are continuous) and $\lim \lambda_n(M) = \lambda(M)$ exists $\forall M \subseteq N$, then $\lambda_n \to \lambda$ uniformly on $2^N$.

**Proof.** This is a particular case of [6, Lemma 1]. To prove this we have only to note that by Osgood's theorem [5, p. 86], the sequence $\{\lambda_n\}$ is equicontinuous at some point of $2^N$. For completeness we give details.

Since $\{0,1\}$ is a topological group, with discrete topology $(1 + 1 = 0, \mod 2), G = 2^N = \{0,1\}^N$, with product topology, is also a topological group, which we write additively with neutral element $0$. Fix $\epsilon > 0$ and suppose $\lambda_n$'s are equicontinuous at $p \in G$. There exist a $0$-nbd

$$V = \left( \prod_{i=1}^m \xi_i \right) \left( \prod_{j=m+1}^\infty J_j \right),$$

where $\xi_i = \{0\}, 1 \leq i \leq m$, and $J_j = \{0,1\}, m + 1 \leq j < \infty$, such that

$$|\lambda_n(p + V) - \lambda_n(p)| < \epsilon/8, \forall n.$$  

Let $p = (p_1, p_2, \ldots, p_m, p_{m+1}, \ldots)$ and $p' = (p_1, p_2, \ldots, p_m, 0, 0, \ldots)$ and $p'' = (0, 0, \ldots, 0, p_{m+1}, p_{m+2}, \ldots). p = p' + p''$. Fix $v \in V$ and take $v' \in V$ such that $p'' + v' = v$. From (*) we get $|\lambda_n(p' + p'' + v') - \lambda_n(p' + p'')| < \epsilon/8$ and so $|\lambda_n(p') + \lambda_n(v) - \lambda_n(p') - \lambda_n(p'')| < \epsilon/8$ (note $\lambda_n$'s are additive). This gives $|\lambda_n(v) - \lambda_n(p'')| < \epsilon/8, \forall v \in V$. In particular, $|\lambda_n(p'')| < \epsilon/8$. Combining these results we get $|\lambda_n(V)| < \epsilon/4, \forall n$. Since $\forall_0 = \{1, 2, \ldots, m\} \cap 2^N$ (i.e., subsets of $\{1,2,\ldots,m\}$) is finite there exists a positive integer $n_0$ such that $|\lambda_n(A) - \lambda(A)| < \epsilon/4, \forall n \geq n_0$ and $A \in \forall_0$. Take $q \in 2^N, q = (q_1, q_2, \ldots, q_m, q_{m+1}, \ldots)$ and put $q' = (q_1, q_2, \ldots, q_m, 0, 0, \ldots), q'' = (0, 0, \ldots, 0, q_{m+1}, q_{m+2}, \ldots)$. Then $q'' + q' = q$ and $q'' \in V$. For $n \geq n_0$,

$$|\lambda_n(q) - \lambda(q)| \leq |\lambda_n(q') - \lambda(q')| + |\lambda_n(q'') - \lambda(q'')|$$

$$\leq \epsilon/4 + |\lambda_n(q')| + |\lambda(q'')| \leq \epsilon/4 + \epsilon/4 + \epsilon/4 < \epsilon.$$
This proves the result.

A subset \( A \subseteq M_\mu(X, E') \) is said to be uniformly tight if, given \( \varepsilon > 0 \), there exists a compact subset \( K \subseteq X \) such that \( |\mu|(X \setminus K) < \varepsilon, \forall \mu \in A \).

**Lemma 2.** A subset \( A \subseteq M_\mu(X, E') \) is \( \beta_0 \)-equicontinuous iff \( A \) is uniformly tight and norm-bounded.

**Proof.** Let \( A \) be norm-bounded and uniformly tight. Put \( \alpha_0 = \sup \{ ||\mu|| : \mu \in A \} \). Since \( \beta_0 \)-topology is the finest locally convex topology, coinciding with compact-open topology on norm-bounded subsets of \( C_b(X, E) \), it is enough to prove that for any \( k > 0 \) there exists a compact subset \( K \subseteq X \) and some \( \eta > 0 \) such that

\[
Z = \{ f \in C_b(X, E) : ||f|| \leq k, ||f||_K \leq \eta \}
\]

\[
\subseteq \{ g \in C_b(X, E) : |\mu(g)| \leq 1, \forall \mu \in A \}.
\]

By uniform tightness of \( A \), there exists a compact \( K \subseteq X \) such that \( |\mu|(X \setminus K) < 1/(2k + 1), \forall \mu \in A \). Take \( \eta = 1/2(1 + \alpha_0) \). For an \( f \in Z \) and \( \mu \in A \),

\[
|\mu(f)| \leq \int ||f|| d|\mu| = \int_K ||f|| d|\mu| + \int_{X \setminus K} ||f|| d|\mu|
\]

\[
\leq \alpha_0/2(1 + \alpha_0) + k/(2k + 1) \leq 1.
\]

This proves \( A \) is \( \beta_0 \)-equicontinuous.

Conversely, if \( A \subseteq M_\mu(X, E') \) is \( \beta_0 \)-equicontinuous then \( A \) is norm-bounded, since \( \beta_0 \leq ||\cdot|| \) on \( C_b(X, E) \). Fix \( \varepsilon > 0 \). There exists a scalar-valued function \( \varphi \) on \( X \) such that

\[
\{ f \in C_b(X, E) : ||f\varphi|| \leq 1 \} \subseteq \{ g \in C_b(X, E) : |\mu(g)| \leq 1, \forall \mu \in A \}.
\]

Take a compact set \( K \), in \( X \), with the property that \( K \supseteq \{ x \in X : |\varphi(x)| \geq \varepsilon \} \). If \( |\mu|(X \setminus K) > \varepsilon \), for some \( \mu \in A \), then, by using the fact that \( \mu_x \in M_\mu(X) \), \( \forall x \in X \), we get a finite disjoint collection \( \{ C_i \} \) of compact subsets of \( X \setminus K \) and \( \{ x_i \} \subseteq E \), with \( ||x_i|| \leq 1, \forall i \), such that \( |\sum \langle \mu(C_i), x_i \rangle| > \varepsilon \). This means there is a collection \( \{ f_i \} \subseteq C_b(X) \), \( 0 \leq f_i \leq 1, \forall i \), supports of \( f_i \)'s mutually disjoint, \( f_i = 0 \) on \( K \), \( \forall i \), such that \( |\mu(f)| > \varepsilon \), where \( f = \sum f_i \otimes x_i \). Now \( ||f\varphi|| \leq \varepsilon \) implies \( |\mu(f)| \leq \varepsilon \), which is a contradiction. This proves the result.

**Lemma 3.** Let \( A \) be a norm-bounded, relatively countably compact subset of \( (F', \sigma(F', F)) \), where \( F = C_b(X, E) \) and \( F' = M_\mu(X, E') \), and assume that \( X \) is a \( P \)-space. Then \( A \) is equicontinuous on \( (F, \beta_0) \).

**Proof.** First we note that \( \mu \in M_\mu(X) \) implies \( |\mu| \in l^1(X) \), since \( X \) is a \( P \)-space [12, p. 467]. Given \( \varepsilon > 0 \), we prove the existence of a finite subset \( D \subseteq X \) such that \( |\mu|(X \setminus D) < \varepsilon, \forall \mu \in A \). Suppose this is not true. Take a \( \mu_1 \in A \) and a finite set \( C_1 \subseteq X \) such that \( |\mu_1|(X \setminus C_1) < \varepsilon/2 \). We get a \( \mu_2 \in A \) such that \( |\mu_2|(X \setminus C_1) > \varepsilon \). Take a finite subset \( C_2 \) of \( X \), \( C_2 \supseteq C_1 \) such that
\[ |\mu_2|(X \setminus C_2) < \varepsilon/2. \]
Continuing this process we get a sequence \( \{\mu_n\} \subset A \), and an increasing sequence \( \{C_n\} \) of finite subsets of \( X \) such that \( |\mu_n|(X \setminus C_i) < \varepsilon/2 \) for \( i \geq n \) and \( |\mu_n|(X \setminus C_i) \geq \varepsilon \) for \( 1 \leq i < n - 1 \). Putting \( C_0 = \emptyset \) and \( D_i = C_i \setminus C_{i-1} \) (\( i = 1, 2, \ldots \)), we get

\[ |\mu_n|(D_n) = |\mu_n|(C_n \setminus C_{n-1}) = |\mu_n|((X \setminus C_{n-1}) \setminus (X \setminus C_n)) \geq \varepsilon/2. \]

Since \( \{D_n\} \) is a disjoint sequence of finite subsets of \( X \), for every \( n \), there exists a finite partition \( \{A_i^{(n)}: 1 \leq i \leq p_n\} \) of \( D_n \) and points \( \{x_i^{(n)}: 1 \leq i \leq p_n\} \) in the closed unit ball of \( E \) such that

\[ \sum_{i=1}^{p_n} \left| \mu_n(x_i^{(n)} \otimes x_{A_i^{(n)}}) \right| \geq \varepsilon/2. \]

Since \( X \) is a \( P \)-space and \( \{A_i^{(n)}: 1 \leq i \leq p_n\} \) is a countable collection of disjoint finite subsets of \( X \), \( \exists \) a disjoint collection of clopen subsets \( \{U_i^{(n)}: 1 \leq i \leq p_n\} \) of \( A_i^{(n)} \) and \( \mu_n(x_i^{(n)} \otimes x_{U_i^{(n)}}) = \mu_n(x_i^{(n)} \otimes x_{U_i^{(n)}}) \), \( \forall n \), and \( \forall i \) (this follows from the regularity of \( x_i^{(n)}, \mu \in \mathcal{M}(X, E') \), \( x \in E \), and the fact that \( X \) is a \( P \)-space). Putting \( f_n = \sum_{i=1}^{p_n} x_i^{(n)} \otimes x_{U_i^{(n)}} \), we get \( |\mu_n(f_n)| > \varepsilon/4 \), \( \forall n \) and \( f_n \in C_b(X, E) \). For any subset \( M \subset N \), \( \sum_{n \in M} f_n = f_M \in C_b(X, E) \) and \( ||f_M|| \leq 1 \) (here we again are using the fact that \( X \) is a \( P \)-space). The space \( H = \{f_M: M \subset N\} \) with topology induced by \( \alpha(F, F') \), contains \( \{f_P: P \subset N, P \text{ finite}\} \) as a dense subset—to prove this, fix \( M \subset N \) and put \( g_m = \sum_{i=1}^{2m} x_i^{(n)} \otimes x_{U_i^{(n)}} \); this gives \( |\mu(f_M - g_m)| \leq |\mu|(||f_M - g_m||) \to 0 \), by the dominated convergence theorem, \( \forall \mu \in F' \). Also \( A \), considered as a set of continuous functions on \( H \), with the topology of pointwise convergence, is relatively countably compact, and so by [7] there exists a subsequence of \( \{\mu_n\} \), which again we denote by \( \{\mu_n\} \), such that \( \{\mu_n\} \) is convergent pointwise on \( H \). Define \( \lambda_n: 2^N \to \mathbb{K} \), \( \lambda_n(M) = \mu_n(f_M) \). It is easy to verify that \( \lambda_n \)'s are countably additive and \( \lim \lambda_n(M) = \lambda(M) \) exists \( \forall M \subset N \). By Lemma 1, \( \lambda_n \to \lambda \) uniformly on \( 2^N \). Choose \( n_0 \in N \) so large that \( ||\lambda_n()|| < \varepsilon/10 \) and \( \forall P \in 2^N, |\lambda_n(P) - \lambda(P)| < \varepsilon/10, \forall n \geq n_0 \). In particular, \( |\lambda_n((n_0)) - \lambda((n_0))| < \varepsilon/10 \), and so \( |\lambda_n((n_0))| < \varepsilon/5 \), i.e., \( |\mu_{n_0}(f_{n_0})| < \varepsilon/5 \). This contradicts \( |\mu_{n_0}(f_{n_0})| > \varepsilon/4, \forall n \). Using Lemma 2, we get the result.

**Example 4.** The condition that \( A \), in Lemma 3, be norm-bounded is essential. Let \( E \) be the subspace of \( l_1 \) over reals, consisting of sequences with only finite number of nonzero components with induced norm. In \( E' = l_{\infty}' \), for every positive integer \( n \), let \( y_n \) have all components 0 except \( n \)th which is equal to \( n \). Put \( A = \{y_n\} \). Now \( y_n \to 0 \) in \( (E', \sigma(E', E)) \), but, being unbounded, is not equiuniform. Thus \( E \) is Mackey but not strongly Mackey. Take \( X = \{x_0\} \), a one-point set. Then \( (C_b(X, E), \beta_0) \) is isometric isomorphic to \( E \). Thus Lemma 3 cannot hold without the assumption of norm-boundedness on \( A \).

**Theorem 5.** If \( X \) is a \( P \)-space and \( E \) a normed space, then \( (C_b(X, E), \beta_0) \) is...
Mackey. If, in addition, E is complete (i.e., E is a Banach space) then 
\((C_b(X,E),\beta_0)\) is strongly Mackey.

Proof. Let A be an absolutely convex, compact subset of \((F',\sigma(F',F))\),
where \(F = (C_b(X,E),\beta_0)\), \(F' = M_t(X,E')\). Since the bounded subsets of
\((C_b(X,E),\beta_0)\) are norm-bounded, the strong topology on \(M_t(X,E')\) is
the norm topology and so A is norm-bounded [8, 5.1, p. 141]. By Lemma 3, A is
equicontinuous. If E is a Banach space, then \(G = (C_b(X,E),\|\cdot\|)\) is also a
Banach space and \(M_t(X,E') \subseteq G'\). Thus if A is a relatively countably
compact subset of \((M_t(X,E'),\sigma(M_t(X,E'))\), \(C_b(X,E))\), then A is a relatively
countably compact subset of \((G',\sigma(G',G))\) and so is norm-bounded. Lemma
3 now gives the result. This completes the proof.

Remark 6. Our proof is different from the usual proof that the function
space be Mackey; the usual proof starts out with “gliding hump” argument
and then uses \(l_\infty\) trick [11]. This theorem generalizes the main result of [11].

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