STRUCTURE THEOREM FOR $A$-COMPACT OPERATORS

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Abstract. A contraction $T$ defined on a complex Hilbert space is called $A$-compact if there exists a nonzero function $f$ analytic in the open unit disc and continuous on the closed disc such that $f(T)$ is a compact operator. In this paper, the factorization of $f$ is used to obtain a structure theorem which describes the spectrum of $T$.

Introduction. A bounded linear operator $T$ on a complex Banach space $X$ is called polynomially compact if there is a nonzero polynomial $p(z)$ such that $p(T)$ is compact. The spectrum of polynomially compact operators $T$ has been completely described by F. Gilfeather [4] by showing that $T$ is a finite sum of power compact operators. A similar problem of describing the spectrum arises in case $p(z)$ is replaced by the uniform limit of polynomials. More explicitly, let $T$ be a contraction defined on a Hilbert space, and $A$ be the uniform closure of polynomials in the supremum norm over $\overline{D}$; $D$ is the open unit disc. The elements of $A$ are analytic functions in $D$ which have continuous extension to $\overline{D}$. For $f(z) = \sum a_n z^n \in A$, $f(T) = \sum a_n T^n$ converges in norm, and $\|f(T)\| \leq \|f\|_{\infty}$. If $f \in H^\infty$, $f_r(z) = f(rz)$ belongs to $A$ for each $r, 0 < r < 1$, and $\lim_{r \to 1-0} f_r(T)$ exists in the norm. Define $f(T) = \lim_{r \to 1-0} f_r(T)$. For $f \in A$, this definition is consistent. The contraction $T$ is called $A$-compact if there exists a nonzero $f \in A$ such that $f(T)$ is a compact operator. The problem is to describe the spectrum of $A$-compact contractions.

Let $T$ be an $A$-compact contraction on a complex Hilbert space. Denote by $J(T)$ the set of functions $f \in A$ such that $f(T)$ is compact. It is easily seen that $J(T)$ is a nonzero closed ideal of $A$, indeed, it is a principal ideal [5, Corollary, p. 88]. Let $Z$ be the set of common zeros of functions in $J(T)$ in $\overline{D}$. Every function which generates $J(T)$ is of the form $f = Fg$ where $F$ is the greatest common divisor of the inner parts of functions in $J(T)$ and $g$ is the outer part of any function which belongs to $A$ and vanishes precisely on $Z \cap \{|z| = 1\}$. A generator $f$ of $J(T)$ is called a minimal function of $T$ if its singularities which lie on the unit circle are contained in $Z$. A minimal function $f$ always exists.

An operator $T$ defined on a Hilbert space $X$ is said to be decomposed if there exists a pair $(X_1, X_2)$ of subspaces of $X$ such that $X$ has unique decomposition in $(X_1, X_2)$ and $X_i$ is invariant under $T$, $i = 1, 2$. A decomposition of $T$ can be obtained by the decomposition theorem [7, p. 419] if $\sigma(T)$ has more than one
connected component. Let $\lambda$ be an isolated point of $\sigma(T)$, and $\Gamma$ be a rectifiable, simple, closed curve enclosing no point of $\sigma(T)$ other than $\lambda$ and containing no point of $\sigma(T)$. Then

\[
E_\lambda = -\frac{1}{2\pi i} \int_{\xi \in \Gamma} (T - \xi I)^{-1} d\xi
\]

is a projection on $X_\lambda = E_\lambda X$ along $X_0 = (I - E_\lambda)X$. The pair $(X_\lambda, X_0)$ decomposes $T$ into $T_\lambda = T/X_\lambda$ and $T_0 = T/X_0$, and the spectra of the two restrictions are $\{\lambda\}$ and $\sigma(T) - \{\lambda\}$, respectively. The terms $X_\lambda$ and $T_\lambda$ will be used afterwards.

A contraction $T$ defined on a Hilbert space $X$ is called completely nonunitary (c.n.u.) if there exists no nonnull reducing subspace on which $T$ is unitary.

1. We now state the structure theorem.

**Theorem.** Let $f$ be a minimal function of a $A$-compact contraction $T$ defined on a complex Hilbert space $X$, and $Z$ be the set of zeros of $f$ in $\overline{D}$. Then (i) $Z \subseteq \sigma(T)$, (ii) $\sigma(f(T)) = f(\sigma(T))$, (iii) $\sigma(T)$ is totally disconnected, (iv) points of $\sigma(T) - Z$ are isolated and are eigenvalues with finite dimensional generalized eigenspaces, and (v) if $\lambda \in Z$ is an isolated point of $\sigma(T)$, then $X_\lambda$ is infinite dimensional, $(T_\lambda - \lambda I)$ is power compact in case $f$ is analytic at $\lambda$; $g_\lambda(T_\lambda)$ is quasinilpotent compact operator in case $\lambda$ is a singularity of $f$ and

\[
g_\lambda(z) = (z - \lambda)\exp(\rho(z + \lambda)/(z - \lambda)), \quad \rho > 0.
\]

**Proof.** (i) By the factorization theorem [5, p. 67], $f$ is uniquely expressible in the form $f(z) = B(z)S(z)F(z)$ where $B(z)$ is a Blaschke product of the form

\[
B(z) = r^p \prod_{n=0}^{\infty} \left( \frac{\alpha_n}{|\alpha_n|} \frac{\alpha_n - z}{1 - \alpha_n \overline{z}} \right)^{p_n},
\]

where $|r| = 1$ and $\prod |\alpha_n|^{p_n}$ converges; $S(z)$ is the singular part of the form

\[
S(z) = \exp\left[ -\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \right],
\]

where $\mu(t)$ is a bounded, nondecreasing function on $(0, 2\pi)$ such that $\mu'(t) = 0$ almost everywhere on $(0, 2\pi]$, and $F(z)$ is the outer function of the form

\[
F(z) = \exp\left[ \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log |f(e^{it})| dt \right].
\]

If possible, let $\lambda \in Z - \sigma(T)$. Then the distance $(\lambda, \sigma(T)) > \delta$ for some $\delta > 0$. Denote by $M = \{z \in \overline{D}; |z - \lambda| < \delta\}$. Let $B_M(z)$ be the subproduct of $B(z)$ which vanishes in $M$, and

\[
S_M(z) = 1 \quad \text{if} \quad M \subset D,
\]

\[
= \exp\left[ \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \right].
\]
where \( \nu(t) \) is a nondecreasing function such that \( \mu(t) = \nu(t) \) if \( e^{it} \in M \) and \( \nu'(t) = 0 \) if \( e^{it} \notin M \). Then the function \( g(z) = B_M(z)S_M(z) \) is analytic and does not vanish in some neighbourhood of \( \sigma(T) \). Hence, \( g(T) \) possesses bounded inverse which implies that \( f(T)g(T)^{-1} \) is compact. It means that \( f/g \in J(T) \), i.e., \( g \) divides (nontrivially) the inner part of \( f \) which contradicts the minimality of \( f \). Hence \( Z \subset \sigma(T) \).

(ii) It is the well-known spectral mapping theorem, and has been proved by Foiaş and Mlak [3, pp. 239–245] for c.n.u. operators. We extend it for general contractions. Let \( T = T_u \oplus T_0 \) be the decomposition of \( T \) into unitary and c.n.u. parts, and let the corresponding decomposition of \( X \) and \( f(T) \) be \( X = X_u \oplus X_0 \) and \( f(T) = f(T_u) \oplus f(T_0) \), respectively. Then, in order to prove (ii), it is enough that \( \sigma(f(T_u)) = f(\sigma(T_u)) \) so that

\[
f(\sigma(T)) = f(\sigma(T_u)) \cup f(\sigma(T_0)) = f(T_u) \cup f(T_0) = f(T).
\]

If \( X_u \) is finite dimensional, it is trivially seen that \( \sigma(f(T_u)) = f(\sigma(T_u)) \). Assume that \( X_u \) is infinite dimensional. It then follows from the spectral representation of unitary operators that \( T_u \) has an infinite dimensional reducing subspace. Therefore, \( f(T_u) \), being the uniform limit of polynomials in \( T_u \), has an infinite dimensional reducing subspace. The compactness of \( f(T_u) \) implies that \( f(T_u) \) is null. So \( \sigma(f(T_u)) = \{0\} \).

For each \( \lambda \in \sigma(T) \) and \( r, 0 < r < 1, f_r(\lambda) \in f(\sigma(T_u)) = f(\sigma(T_u)) \). The subadditivity of commuting operators implies that

\[
|f_r(\lambda)| \leq \text{spectral radius of } f(T_u).
\]

and since \( |f(\lambda)| = \lim_{r \to 1^-} |f_r(\lambda)| \), it follows that \( f(\lambda) = 0 \) for each \( \lambda \in \sigma(T) \), i.e., \( \sigma(f(T_u)) = \{0\} \). Hence \( \sigma(f(T_u)) = f(\sigma(T_u)) \). This proves (ii).

(iii) First, we show that \( \sigma(T) - Z \) contains only isolated points of \( \sigma(T) \). Since \( f(T) \) is compact, and \( \sigma(f(T)) = f(\sigma(T)) \), the only possible accumulation point of \( f(\sigma(T)) \) can be zero. The continuity of \( f \) in \( D \) implies that the possible accumulation points of \( \sigma(T) \) are zeros of \( f \) in \( \overline{D} \), i.e., \( \sigma(T) - Z \) contains only isolated points.

To prove (iii), it is enough to show that \( Z \) is totally disconnected. If possible, let \( Z \) have a connected component which has more than one point. Then either \( Z \) has a point of accumulation in \( D \) or contains an arc of the unit circle. In either case \( f = 0 \), a contradiction. The earlier case follows from the analyticity of \( f \) in \( D \) and the later case follows from a well-known result of F. and M. Riesz about bounded analytic functions.

(iv) Let \( \lambda \in \sigma(T) - Z \). By (iii), \( \lambda \) is an isolated point of \( \sigma(T) - Z \). By the definition of minimal function, \( f \) can be assumed to be analytic in some neighbourhood of \( \lambda \). Let \( X_\lambda \) and \( T_\lambda \) be as defined by (•). Since \( f(\lambda) \neq 0, \sigma(T_\lambda) = \{\lambda\} \) and \( \sigma(f(T_\lambda)) = f(\sigma(T_\lambda)) \), it follows that \( 0 \notin \sigma(f(T_\lambda)) \), or equivalently, \( f(T_\lambda) \) has bounded inverse. Moreover, \( f(T_\lambda) \) is compact, therefore, its domain \( X_\lambda \) must necessarily be finite dimensional, say \( n > 0 \). Since \( T_\lambda \) has a matrix representation with respect to each basis of \( X_\lambda \), hence \( (T_\lambda - \lambda I)^n = 0 \) and
This completes (iv).

(v) Let $\lambda \in \mathbb{Z}$ be a given isolated point of $\sigma(T)$, $(X_{\lambda}, X_0)$ be the decomposition of $X$ obtained by (•), and $T = T_{\lambda} + T_0$. If $X_{\lambda}$ is finite dimensional, then $T_{\lambda}$ is compact, and therefore for each $g \in A$, $g(T)$ is compact, if and only if, $g(T_0)$ is compact. In particular, if

$$g(z) = f(z)(z - \lambda)^{-1}\exp(-\rho(z + \lambda)/(z - \lambda)), \quad \rho > 0,$$

then $g(T_0)$ exists because $\lambda \notin \sigma(T_0)$. Moreover $g(T_0)$ is compact which implies that $g \in J(T)$. This contradicts that $f$ is a minimal function of $T$.

Now, consider the case that $f(z)$ is analytic in some neighbourhood of $\lambda$. The set $J(T_{\lambda}) = \{g \in A: g(T_{\lambda}) \text{ is compact}\}$ is a nonzero closed ideal in $A$. An application of (i) and (iv) for $J_{\lambda}$ implies that $\lambda$ is the only common zero of functions in $J(T_{\lambda})$ in $D$. It means that $J(T_{\lambda})$ is a primary ideal in $A$ and is generated by $(z - \lambda)^k$ for some positive integer $k$ [5, Corollary, p. 88] which proves that $(T_{\lambda} - \lambda I)^k$ is compact, i.e., $(T_{\lambda} - \lambda I)$ is power compact.

Next, consider that $\lambda$ is a singularity of $f$. In this case, the ideal $J(T_{\lambda})$ is generated by $g_{\lambda}(z) = (z - \lambda)\exp(\rho(z + \lambda)/(z - \lambda)), \rho > 0$, which implies that $g_{\lambda}(T_{\lambda})$ is compact. The quasinilpotency of $g_{\lambda}(1)$ follows from the fact that $\sigma(g_{\lambda}(T_{\lambda})) = g_{\lambda}(\sigma(T_{\lambda})) = g_{\lambda}(\{\lambda\}) = \{0\}$. It completes the proof of the theorem.

2. We give below two corollaries and an example of an $A$-compact operator which is not polynomially compact.

**Example.** Let $S(z) = \exp((z + 1)/(1 - z))$ and $f(z) = (z - 1)S(z)$. Consider the contraction $T$ on the Hilbert space $H(S) = H^2 \ominus SH^2$ defined by

$$T: g(z) \rightarrow zg(z) - S(z)g(0)$$

where

$$g(0) = \langle g(z), (S(z) - S(0))/z \rangle_{H(S)}.$$

Then it follows from [6, Proposition 3.2, p. 119] that $S(T) = 0$ and from [6, Theorem 5.1, p. 126] that $\sigma(T) = \{1\}$. Now it is clear from [4, Theorem 1] that $T$ cannot be polynomially compact. However, $T$ is $A$-compact because $f(T) = 0$.

**Corollary 1.** If $T$ is $A$-compact and $\sigma(T) \subseteq D$, then $T$ is polynomially compact.

**Proof.** Follows immediately from (i).

Sz.-Nagy and Foiaş have denoted by $C_0$ the class of those c.n.u. operators $T$ for which there exists a nonzero $u \in H^\infty$ such that $u(T) = 0$, and have described the spectrum of such operators, viz., [6, Theorem 5.1, p. 126]. We prove this result for the related class $\tilde{C}_0(A)$ defined as

$$\tilde{C}_0(A) = \{T: \|T\| \leq 1 \text{ and for some nonzero } f \in A, f(T) \text{ is a quasi-nilpotent compact operator}\}.$$
Corollary 2. If $T \in C_0(A)$, then $\sigma(T) = Z$, the set of zeros of the minimal function of $T$.

Proof. Let $f$ be a minimal function of $T$ for which $f(T)$ is compact and quasi-nilpotent. By (ii), $f(\sigma(T)) = \sigma(f(T)) = \{0\}$ which implies that $\sigma(T) \subseteq Z$. Adding to (i), $\sigma(T) = Z$.

References