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STRUCTURE THEOREM FOR A-COMPACT OPERATORS

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ABSTRACT. A contraction T defined on a complex Hilbert space is called A-compact if there exists a nonzero function f analytic in the open unit disc and continuous on the closed disc such that $f(T)$ is a compact operator. In this paper, the factorization of f is used to obtain a structure theorem which describes the spectrum of T.

Introduction. A bounded linear operator T on a complex Banach space X is called polynomially compact if there is a nonzero polynomial $p(z)$ such that $p(T)$ is compact. The spectrum of polynomially compact operators T has been completely described by F. Gilfeather [4] by showing that T is a finite sum of power compact operators. A similar problem of describing the spectrum arises in case $p(z)$ is replaced by the uniform limit of polynomials. More explicitly, let T be a contraction defined on a Hilbert space, and A be the uniform closure of polynomials in the supremum norm over $\overline{D}$; D is the open unit disc. The elements of A are analytic functions in D which have continuous extension to $\overline{D}$. For $f(z) = \sum a_n z^n \in A$, $f(T) = \sum a_n T^n$ converges in norm, and $\|f(T)\| \leqslant \|f\|_\infty$. If $f \in H^\infty$, $f_r(z) = f(rz)$ belongs to A for each r, $0 < r < 1$, and $\lim_{r\to 1-0} f_r(T)$ exists in the norm. Define $f(T) = \lim_{r\to 1-0} f_r(T)$. For $f \in A$, this definition is consistent. The contraction T is called *A-compact* if there exists a nonzero $f \in A$ such that $f(T)$ is a compact operator. The problem is to describe the spectrum of A-compact contractions.

Let T be an A-compact contraction on a complex Hilbert space. Denote by $J(T)$ the set of functions $f \in A$ such that $f(T)$ is compact. It is easily seen that $J(T)$ is a nonzero closed ideal of A, indeed, it is a principal ideal [5, Corollary, p. 88]. Let Z be the set of common zeros of functions in $J(T)$ in $\overline{D}$. Every function which generates $J(T)$ is of the form $f = Fg$ where F is the greatest common divisor of the inner parts of functions in $J(T)$ and g is the outer part of any function which belongs to A and vanishes precisely on $Z \cap \{|z| = 1\}$. A generator f of $J(T)$ is called a *minimal function* of T if its singularities which lie on the unit circle are contained in Z. A minimal function f always exists.

An operator T defined on a Hilbert space X is said to be *decomposed* if there exists a pair (X_1, X_2) of subspaces of X such that X has unique decomposition in (X_1, X_2) and X_i is invariant under T, $i = 1, 2$. A decomposition of T can be obtained by the decomposition theorem [7, p. 419] if $\sigma(T)$ has more than one

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connected component. Let λ be an isolated point of $\sigma(T)$, and Γ be a rectifiable, simple, closed curve enclosing no point of $\sigma(T)$ other than λ and containing no point of $\sigma(T)$. Then

$$(*) \qquad E_\lambda = -\frac{1}{2\pi i}\int_{\xi\in\Gamma}(T-\xi I)^{-1}\,d\xi$$

is a projection on $X_\lambda = E_\lambda X$ along $X_0 = (I - E_\lambda)X$. The pair (X_λ, X_0) decomposes T into $T_\lambda = T/X_\lambda$ and $T_0 = T/X_0$, and the spectra of the two restrictions are $\{\lambda\}$ and $\sigma(T) - \{\lambda\}$, respectively. The terms X_λ and T_λ will be used afterwards.

A contraction T defined on a Hilbert space X is called completely nonunitary (c.n.u.) if there exists no nonnull reducing subspace on which T is unitary.

1. We now state the structure theorem.

THEOREM. *Let f be a minimal function of a A-compact contraction T defined on a complex Hilbert space X, and Z be the set of zeros of f in $\overline{D}$. Then* (i) $Z \subseteq \sigma(T)$, (ii) $\sigma(f(T)) = f(\sigma(T))$, (iii) $\sigma(T)$ *is totally disconnected,* (iv) *points of $\sigma(T) - Z$ are isolated and are eigenvalues with finite dimensional generalized eigenspaces, and* (v) *if $\lambda \in Z$ is an isolated point of $\sigma(T)$, then X_λ is infinite dimensional, $(T_\lambda - \lambda I)$ is power compact in case f is analytic at λ; $g_\lambda(T_\lambda)$ is quasinilpotent compact operator in case λ is a singularity of f and*

$$g_\lambda(z) = (z-\lambda)\exp(\rho(z+\lambda)/(z-\lambda)), \qquad \rho > 0.$$

PROOF. (i) By the factorization theorem [5, p. 67], f is uniquely expressible in the form $f(z) = B(z)S(z)F(z)$ where $B(z)$ is a Blaschke product of the form

$$B(z) = rz^p \prod_i^\infty \left(\frac{\overline{\alpha}_n}{|\alpha_n|}\,\frac{\alpha_n - z}{1-\overline{\alpha}_n z}\right)^{p_n},$$

where $|r| = 1$ and $\prod |\alpha_n|^{p_n}$ converges; $S(z)$ is the singular part of the form

$$S(z) = \exp\left[-\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\,d\mu(t)\right],$$

where $\mu(t)$ is a bounded, nondecreasing function on $(0, 2\pi]$ such that $\mu'(t) = 0$ almost everywhere on $(0, 2\pi]$, and $F(z)$ is the outer function of the form

$$F(z) = \exp\left[\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log|f(e^{it})|dt\right].$$

If possible, let $\lambda \in Z - \sigma(T)$. Then the distance $(\lambda, \sigma(T)) > \delta$ for some $\delta > 0$. Denote by $M = \{z \in \overline{D}\colon |z-\lambda| \leqslant \delta\}$. Let $B_M(z)$ be the subproduct of $B(z)$ which vanishes in M, and

$$\begin{aligned} S_M(z) &= 1 \quad \text{if } M \subset D, \\ &= \exp\left[-\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\,d\nu(t)\right], \end{aligned}$$

where $\nu(t)$ is a nondecreasing function such that $\mu(t) = \nu(t)$ if $e^{it} \in M$ and $\nu'(t) = 0$ if $e^{it} \notin M$. Then the function $g(z) = B_M(z)S_M(z)$ is analytic and does not vanish in some neighbourhood of $\sigma(T)$. Hence, $g(T)$ possesses bounded inverse which implies that $f(T)g(T)^{-1}$ is compact. It means that $f/g \in J(T)$, i.e., g divides (nontrivially) the inner part of f which contradicts the minimality of f. Hence $Z \subset \sigma(T)$.

(ii) It is the well-known spectral mapping theorem, and has been proved by Foiaş and Mlak [3, pp. 239–245] for c.n.u. operators. We extend it for general contractions. Let $T = T_u \oplus T_0$ be the decomposition of T into unitary and c.n.u. parts, and let the corresponding decomposition of X and $f(T)$ be $X = X_u \oplus X_0$ and $f(T) = f(T_u) \oplus f(T_0)$, respectively. Then, in order to prove (ii), it is enough that $\sigma(f(T_u)) = f(\sigma(T_u))$ so that

$$f(\sigma(T)) = f(\sigma(T_u)) \cup f(\sigma(T_0)) = \sigma(f(T_u)) \cup \sigma(f(T_0)) = \sigma(f(T)).$$

If X_u is finite dimensional, it is trivially seen that $\sigma(f(T_u)) = f(\sigma(T_u))$. Assume that X_u is infinite dimensional. It then follows from the spectral representation of unitary operators that T_u has an infinite dimensional reducing subspace. Therefore, $f(T_u)$, being the uniform limit of polynomials in T_u, has an infinite dimensional reducing subspace. The compactness of $f(T_u)$ implies that $f(T_u)$ is null. So $\sigma(f(T_u)) = \{0\}$.

For each $\lambda \in \sigma(T)$ and r, $0 < r < 1$, $f_r(\lambda) \in f_r(\sigma(T_u)) = \sigma(f_r(T_u))$. The subadditivity of commuting operators implies that

$$\begin{aligned} |f_r(\lambda)| &\leqslant \text{spectral radius of } f_r(T_u) \\ &\leqslant \text{spectral radius of } [f_r(T_u) - f(T_u)] + \text{spectral radius of } f(T_u) \\ &\leqslant \|f_r(T_u) - f(T_u)\| + 0, \end{aligned}$$

and since $|f(\lambda)| = \lim_{r \to 1-0} |f_r(\lambda)|$, it follows that $f(\lambda) = 0$ for each $\lambda \in \sigma(T)$, i.e., $f(\sigma(T_u)) = \{0\}$. Hence $f(\sigma(T_u)) = \sigma(f(T_u))$. This proves (ii).

(iii) First, we show that $\sigma(T) - Z$ contains only isolated points of $\sigma(T)$. Since $f(T)$ is compact, and $\sigma(f(T)) = f(\sigma(T))$, the only possible accumulation point of $f(\sigma(T))$ can be zero. The continuity of f in $\overline{D}$ implies that the possible accumulation points of $\sigma(T)$ are zeros of f in $\overline{D}$, i.e., $\sigma(T) - Z$ contains only isolated points.

To prove (iii), it is enough to show that Z is totally disconnected. If possible, let Z have a connected component which has more than one point. Then either Z has a point of accumulation in D or contains an arc of the unit circle. In either case $f \equiv 0$, a contradiction. The earlier case follows from the analyticity of f in D and the later case follows from a well-known result of F. and M. Riesz about bounded analytic functions.

(iv) Let $\lambda \in \sigma(T) - Z$. By (iii), λ is an isolated point of $\sigma(T) - Z$. By the definition of minimal function, f can be assumed to be analytic in some neighbourhood of λ. Let X_λ and T_λ be as defined by (*). Since $f(\lambda) \neq 0$, $\sigma(T_\lambda) = \{\lambda\}$ and $\sigma(f(T_\lambda)) = f(\sigma(T_\lambda))$, it follows that $0 \notin \sigma(f(T_\lambda))$, or equivalently, $f(T_\lambda)$ has bounded inverse. Moreover, $f(T_\lambda)$ is compact, therefore, its domain X_λ must necessarily be finite dimensional, say $n > 0$. Since T_λ has a matrix representation with respect to each basis of X_λ, hence $(T_\lambda - \lambda I)^n = 0$ and

$$X_\lambda = \{x \in X\colon (T - \lambda I)^n x = 0\}.$$

This completes (iv).

(v) Let $\lambda \in Z$ be a given isolated point of $\sigma(T)$, (X_λ, X_0) be the decomposition of X obtained by (*), and $T = T_\lambda + T_0$. If X_λ is finite dimensional, then T_λ is compact, and therefore for each $g \in A$, $g(T)$ is compact, if and only if, $g(T_0)$ is compact. In particular, if

$$g(z) = f(z)(z - \lambda)^{-1}\exp(-\rho(z + \lambda)/(z - \lambda)), \qquad \rho > 0,$$

then $g(T_0)$ exists because $\lambda \notin \sigma(T_0)$. Moreover $g(T_0)$ is compact which implies that $g \in J(T)$. This contradicts that f is a minimal function of T.

Now, consider the case that $f(z)$ is analytic in some neighbourhood of λ. The set $J(T_\lambda) = \{g \in A\colon g(T_\lambda)$ is compact$\}$ is a nonzero closed ideal in A. An application of (i) and (iv) for T_λ implies that λ is the only common zero of functions in $J(T_\lambda)$ in $\overline{D}$. It means that $J(T_\lambda)$ is a primary ideal in A and is generated by $(z - \lambda)^k$ for some positive integer k [5, Corollary, p. 88] which proves that $(T_\lambda - \lambda I)^k$ is compact, i.e., $(T_\lambda - \lambda I)$ is power compact.

Next, consider that λ is a singularity of f. In this case, the ideal $J(T_\lambda)$ is generated by $g_\lambda(z) = (z - \lambda)\exp(\rho(z + \lambda)/(z - \lambda))$, $\rho > 0$, which implies that $g_\lambda(T_\lambda)$ is compact. The quasinilpotency of $g_\lambda(T_\lambda)$ follows from the fact that $\sigma(g_\lambda(T_\lambda)) = g_\lambda(\sigma(T_\lambda)) = g_\lambda(\{\lambda\}) = \{0\}$. It completes the proof of the theorem.

2. We give below two corollaries and an example of an A-compact operator which is not polynomially compact.

EXAMPLE. Let $S(z) = \exp((z + 1)/(1 - z))$ and $f(z) = (z - 1)S(z)$. Consider the contraction T on the Hilbert space $H(S) = H^2 \ominus SH^2$ defined by

$$T\colon g(z) \to zg(z) - S(z)\hat{g}(0)$$

where

$$\hat{g}(0) = \langle g(z), (S(z) - S(0))/z\rangle_{H(S)}.$$

Then it follows from [6, Proposition 3.2, p. 119] that $S(T) = 0$ and from [6, Theorem 5.1, p. 126] that $\sigma(T) = \{1\}$. Now it is clear from [4, Theorem 1] that T cannot be polynomially compact. However, T is A-compact because $f(T) = 0$.

COROLLARY 1. *If T is A-compact and $\sigma(T) \subset D$, then T is polynomially compact.*

PROOF. Follows immediately from (i).

Sz.-Nagy and Foiaş have denoted by C_0 the class of those c.n.u. operators T for which there exists a nonzero $u \in H^\infty$ such that $u(T) = 0$, and have described the spectrum of such operators, viz., [6, Theorem 5.1, p. 126]. We prove this result for the related class $\hat{C}_0(A)$ defined as

$$\hat{C}_0(A) = \{T\colon \|T\| \leqslant 1 \text{ and for some nonzero } f \in A, \\ f(T) \text{ is a quasi-nilpotent compact operator}\}.$$

COROLLARY 2. *If $T \in \hat{C}_0(A)$, then $\sigma(T) = Z$, the set of zeros of the minimal function of T.*

PROOF. Let f be a minimal function of T for which $f(T)$ is compact and quasi-nilpotent. By (ii), $f(\sigma(T)) = \sigma(f(T)) = \{0\}$ which implies that $\sigma(T) \subseteq Z$. Adding to (i), $\sigma(T) = Z$.

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