ON DERIVATION ALGEBRAS OF MALCEV ALGEBRAS

ERNEST L. STITZINGER

Abstract. It is shown that if $A$ is a Malcev algebra over a field of characteristic 0, then $A$ is semisimple if and only if the derivation algebra $\mathfrak{D}(A)$ is semisimple. It is then shown that $A$ is semisimple if and only if $A^* = \mathfrak{L}(A) + \mathfrak{D}(A)$ is semisimple, where $\mathfrak{L}(A)$ is the Lie multiplication algebra of $A$.

Let $A$ be a nonassociative algebra over a field of characteristic 0 and let $\mathfrak{D}(A)$ be the derivation algebra of $A$. For certain classes of algebras (Lie, Jordan, alternative), there are results linking the semisimplicity of $A$ to that of $\mathfrak{D}(A)$. Although some results of a related nature are available for Malcev algebras, it has not yet been shown that the semisimplicity of $A$ and $\mathfrak{D}(A)$ are equivalent. It is the purpose of the present note to obtain this result and several related ones.

All algebras discussed here will be finite dimensional over a field of characteristic 0.

Theorem 1. Let $A$ be a Malcev algebra over a field $F$ of characteristic 0. Then $A$ is semisimple if and only if $\mathfrak{D}(A)$ is semisimple.

Proof. Suppose that $\mathfrak{D}(A)$ is semisimple. By [11, Theorem 1] the radical $R(A)$ of $A$ is contained in the center $Z(A)$ of $A$. Then by [10, Theorem 1] $A$ is the direct sum of $R(A)$ and a maximal semisimple subalgebra $S(A)$ of $A$. Then $\mathfrak{D}(A)$ is the direct sum of ideals $\mathfrak{D}_1$ and $\mathfrak{D}_2$ where

$$\mathfrak{D}_1 = \{D \in \mathfrak{D}(A); D: S(A) \to S(A), D: R(A) \to 0\}$$

and

$$\mathfrak{D}_2 = \{D \in \mathfrak{D}(A); D: S(A) \to 0\}$$

(see the proof of [11, Theorem 1]). Suppose that $R(A) \neq 0$. Let $T$ be the projection of $A$ onto $R(A)$ with null space $S(A)$. Since $0 \neq R(A) \subseteq Z(A)$, $T \in \mathfrak{D}_2$ and $T \neq 0$. Let $T_1 \in \mathfrak{D}_2$, $x \in R(A)$ and $y \in S(A)$. Then, since $R(A)$ is $\mathfrak{D}(A)$-invariant by [3, Theorem 14],

Received by the editors December 17, 1975 and, in revised form, June 8, 1976.


© American Mathematical Society 1977

31
\[(x + y)[T, T'] = (x + y)TT' - (x + y)T'T = xT' - xT' = 0.\]

Hence \(T \in Z(\mathfrak{D}_2)\) and since \(\mathfrak{D}_1\) and \(\mathfrak{D}_2\) are ideals in \(\mathfrak{D}(A)\), \(T \in Z(\mathfrak{D}(A))\).

This contradicts the semisimplicity of \(\mathfrak{D}(A)\), hence \(R(A) = 0\).

Conversely let \(K\) be an algebraic closure of \(F\). Then \(\mathfrak{D}(A_K) \simeq \mathfrak{D}(A)_K\) (see [6, p. 233]). Semisimplicity of a Malcev algebra is equivalent to nondegeneracy of the Killing form [5, Theorem A] and the latter is preserved under extension of the base field, hence \(A_K\) is semisimple. By [5, Korollar 1], \(A_K\) is the direct sum of simple ideals. Each of these simple ideals is either a simple Lie algebra or is a 7 dimensional algebra obtained from a Cayley algebra as in [8, pp. 433–435]. This follows from combined results of Sagle [9] and Loos [5, Theorem B]. In either case, the simple algebra has simple derivation algebra.

In particular, the non-Lie simple algebra has the exceptional simple Lie algebra \(G_2\) for its derivation algebra [8, p. 455]. It follows that \(\mathfrak{D}(A_K)\) is semisimple, hence \(\mathfrak{D}(A)\) is also.

In order to extend a classical result of Leger and Togo [4] to general algebras, Ravisankar [6] considered \(A^* = \mathfrak{L}(A) + \mathfrak{D}(A)\) where \(\mathfrak{L}(A)\) is the Lie multiplication algebra of \(A\). \(\mathfrak{L}(A)\) is an ideal of \(A^*\) [6, p. 225]. We now show that the result of Theorem 1 holds with \(\mathfrak{D}(A)\) replaced by \(A^*\). In order to do this we need the following result.

**Lemma.** Let \(A\) be a Malcev algebra over a field of characteristic 0. Then \(\mathfrak{L}(A)\) is semisimple or 0 if and only if \(R(A) \subseteq Z(A)\).

**Proof.** If \(R(A) \subseteq Z(A)\), then \(R(A)\) is complemented by a semisimple subalgebra \(S(A)\) by [10, Theorem 1]. If \(S(A) = 0\), then \(\mathfrak{L}(A) = 0\). Suppose that \(S(A) \neq 0\). Then \(S(A)\) is \(\mathfrak{L}(A)\) invariant since \(R(A) \subseteq Z(A)\), and the restriction mapping is an isomorphism from \(\mathfrak{L}(A)\) to \(\mathfrak{L}(S(A))\). Since \(S(A)\) is semisimple, \(\mathfrak{L}(S(A))\) is also [8, Corollary 7.3].

Conversely if \(\mathfrak{L}(A)\) is semisimple, then, using a theorem of Weyl [1, Théorème 2, p. 75], \(Z(A)\) is complemented by an ideal \(B\) of \(A\). Clearly \(Z(B) = 0\) and \(\mathfrak{L}(A) \simeq \mathfrak{L}(B)\). Hence \(B\) is semisimple by [8, Theorem 7.2] and the result holds. If \(\mathfrak{L}(A) = 0\), then \(Z(A) = A\).

**Theorem 2.** Let \(A\) be a Malcev algebra over a field of characteristic 0. Let \(A^* = \mathfrak{L}(A) + \mathfrak{D}(A)\). Then \(A^*\) is semisimple if and only if \(A\) is semisimple.

**Proof.** If \(A\) is semisimple, then \(\mathfrak{D}(A)\) and \(\mathfrak{L}(A)\) are semisimple. Then \(A^*/\mathfrak{L}(A) \simeq \mathfrak{D}(A)/\mathfrak{D}(A) \cap \mathfrak{L}(A)\) is semisimple by [1, Corollaire 2, p. 76]. Since \(R(A^*)\) projects onto the radical of \(A^*/\mathfrak{L}(A)\) [1, Corollaire 3, p. 76], \(R(A^*) \subseteq \mathfrak{L}(A)\). Hence \(R(A^*) = 0\) and \(A^*\) is semisimple.

Conversely, if \(A^*\) is semisimple, then \(\mathfrak{L}(A)\) is semisimple or \(\mathfrak{L}(A) = 0\), hence \(R(A) \subseteq Z(A)\) and \(R(A)\) is complemented by a semisimple subalgebra \(S(A)\). Suppose \(R(A) \neq 0\). By [11, Theorem 1], \(\mathfrak{D}(A)\) acts completely reducibly on \(A\), hence \(\mathfrak{D}(A) = S \oplus Z\) where \(S\) is a semisimple subalgebra of \(\mathfrak{D}(A)\) and \(Z\) is the center of \(\mathfrak{D}(A)\). Since \(A^*/\mathfrak{L}(A)\) is semisimple, \(Z \subseteq \mathfrak{L}(A)\). Let \(T\) be the projection of \(A\) on \(R(A)\) with null space \(S(A)\). Then \(T \in \mathfrak{D}(A)\). Let \(\mathfrak{D}_1\) and \(\mathfrak{D}_2\).
be as in the proof of Theorem 1. Then both $R(A)$ and $S(A)$ are invariant under $\mathfrak{D}_1 \oplus \mathfrak{D}_2 = \mathfrak{D}(A)$, hence $T$ commutes with each element of $\mathfrak{D}(A)$. Therefore $T \in Z \subseteq \mathfrak{L}(A)$. Since $R(A) \subseteq Z(A)$, $\mathfrak{L}(A)$ annihilates $R(A)$. Hence $T = 0$ and $R(A) = 0$.

**References**


Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27607