ON A PROBLEM OF BRUCKNER AND CEDER
CONCERNING
THE SUM OF DARBOUX FUNCTIONS

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Abstract. The main purpose of this paper is to show that for some continuous function $f$ and any preassigned, countable and dense set $D$ of real numbers there exists a measurable function $d$ which takes on every real value in every interval such that the range of $f + d$ is $D$.

A real valued function $f$ defined on an interval $I$ is said to have the intermediate value property if whenever $x_1$ and $x_2$ are in $I$, and $y$ is any number between $f(x_1)$ and $f(x_2)$, there is a number $x_3$ between $x_1$ and $x_2$ such that $f(x_3) = y$. Because of Darboux's work on this property, one now usually calls a function having the intermediate value property a Darboux function. A function $f$ is Darboux if and only if $f$ maps any connected subset of $I$ onto a connected set.

Since the sum of Darboux functions may fail to be a Darboux function, some mathematicians have examined how badly it can fail. Bruckner and Ceder [1] have recently shown that corresponding to each Darboux function $f$ which is not constant on any subinterval, there exists a function $d$ which takes on every real value in every subinterval of $I$ such that the range of $f + d$ is a preassigned countable and dense set. Certainly $d$ is a Darboux function and the range of $f + d$ is totally disconnected.

The technique used by Bruckner and Ceder in constructing the function $d$ does not imply the measurability of $d$. Indeed Bruckner and Ceder have shown that $d$ must be nonmeasurable whenever $f$ is absolutely continuous. They asked whether $d$ had to be nonmeasurable if we weaken absolute continuity of $f$ to continuity of $f$.

This paper answers this question affirmatively for certain continuous functions $f$. This is a consequence of the following.

Theorem. Let $g$ be a Darboux function which is not constant on any subinterval of its domain $I$ such that the set

$$A = \{a: a \in \mathbb{R}, g^{-1}(a) \text{ is perfect}\}$$

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is dense in \( g(I) \). Then there exists a homeomorphism \( h \) from \( I \) onto itself such that for every countable, dense set \( D \subset \mathbb{R} \) there exists a function \( d \) which takes on every real value in every subinterval of \( I \) such that \( D \) is the range of \( (g \circ h) + d \) and \((g \circ h) + d \) is constant almost everywhere.

It is known that there exist continuous functions \( g \) which are not constant on any interval such that \( A = g(I) \). See for example [4]. (The properties of the set \( A \) for a Darboux function \( g \) are discussed in [2] and [3].) Let \( g \) be such a function. Now define \( f \) by \( f = g \circ h \) where \( h \) is the homeomorphism of the Theorem above. Then \( f \) is continuous and from this Theorem the range of \( f + d \) is \( D \). By the Theorem \( f + d \) is constant almost everywhere. Thus \( d \) is measurable.

In order to prove the Theorem we shall use the type of construction suggested by Theorem 1 of [1]. But first we need the following.

**Lemma.** Suppose \( F \subset I \), and for every interval \( J \subset I \) there exists a perfect set \( P \subset F \cap J \). Then there exists a homeomorphism \( \chi \) from \( I \) onto itself such that \( \chi(F) \) is measurable and \( |\chi(F)| = |I| \).

**Proof.** The Lemma is trivial if \( F = I \). Suppose that \( I \setminus F \neq \emptyset \). Let \( P^0 \) be a perfect subset of \( F \). Choose an increasing homeomorphism \( h_1 \) from \( I \) onto itself so that \( |h_1(P^0)| > \frac{1}{2}|I| \). Let \( \{(a_i, b_i)\} \) be a sequence which may be finite, of disjoint open intervals, such that \( I \setminus \{(a) \cup (b) \cup h_1(P^0)\} = \bigcup_i (a_i, b_i) \). Choose a sequence \( \{I_i\} \) of nonoverlapping intervals so that \( n \geq 1 \implies h_2^n(x) = x \), and

\[
|I_i| < \frac{1}{2}|I|.
\]

Next choose for each \( i \) a perfect set \( P_i \subset h_i^{-1}(I_i) \cap F \). Next choose a strictly increasing continuous function \( h_2 \) from \( I \) onto itself such that

\[
x \in h_1(P_0) \implies h_2(x) = x,
\]

\[
h_2(I_i) = I_i
\]

and \( |h_2(h_1(P)))| > \frac{1}{2}|I| \).

By (2) \( x \in P^0 \implies h_1(x) = h_2(h_1(x)) \). Now put \( P^1 = P^0 \cup \bigcup P_i \). Then \( P^1 \) is a perfect set, \( P^0 \subset P^1 \subset F \), \( |h_2(h_1(P^1)))| > \frac{3}{2}|I| \) and by (2), (3) and (1), \( |x - h_2(x)| < \frac{1}{2}|I| \).

Using this method we can define by induction the sequences \( \{P^n\} \) of perfect sets and \( \{h_n\} \) of strictly increasing continuous functions from \( I \) onto itself such that

\[
x = h_2(x) = h_3(x) = \cdots \quad \text{for } x \in h_1(P^0),
\]

\[
x = h_3(x) = h_4(x) = \cdots \quad \text{for } x \in h_2(h_1(P^1))
\]

and generally,

\[
x = h_{n+1}(x) = h_{n+2}(x) = \cdots \quad \text{for } x \in h_n(h_{n-1}(\cdots h_1(P^{n-1}) \cdots)),
\]
(5) \[|x - h_n(x)| < (1/2^n - 1)|I|,\]

(6) \[|h_n(h_{n-1}(\cdots h_1(P^{n-1}) \cdots))| > (2^n - 1/2^n)|I|.\]

By (5) we have \[|h_n(x) - h_{n+1}(h_n(x))| < (1/2^n)|I|.\] Hence

\[|h_n(h_{n-1}(\cdots h_1(x) \cdots)) - h_{n+1}(h_n(\cdots h_1(x) \cdots))| < (1/2^n)|I|.\]

By this inequality the sequence of superpositions \(\{h_n(h_{n-1}(\cdots h_1(x) \cdots))\}\) converges uniformly to a continuous function. Denote this limit function by \(\chi\). All \(h_n\) are strictly increasing so \(\chi\) is increasing. By (4),

(7) \[\chi(x) = h_n(h_{n-1}(\cdots h_1(x) \cdots)) \quad \text{for} \quad x \in P^{n-1}.\]

Hence \(\chi\) is strictly increasing on \(P^{n-1}\) and also on the set \(\bigcup_{n=1}^\infty P^n\) which is dense in \(I\). Thus \(\chi\) is strictly increasing. By (7)

\[\chi(P^{n-1}) = h_n(h_{n-1}(\cdots h_1(P^{n-1}) \cdots)).\]

Hence, since \(\chi(P^{n-1}) \subset \chi(F) \subset I\) by (6) we have \[|\chi(E)| > ((2^n - 1)/2^n)|I|.\] Hence \[|\chi(F)| = |I|\].

This completes the proof of the Lemma.

**Proof of the Theorem.** Let \(\{I_n\}\) be a sequence of all the open intervals with rational endpoints contained in \(I\). Since \(g(I)\) is a subinterval of \(g(I)\) and \(A\) is dense in \(g(I)\), there exists a point \(\alpha_i \in A \cap g(I_i)\). Let \(F_i\) be a perfect set such that \(F_i \subset I_i \cap g^{-1}(\alpha_i)\) and \(F_i\) is nowhere dense in \(g^{-1}(\alpha_i)\). Suppose we have chosen for each \(i = 1, 2, \ldots, k\) perfect sets \(F_i\) and numbers \(\alpha_i\) such that \(F_i \subset I_i\), \(g(F_i) = \alpha_i \in A\), \(F_i\) is nowhere dense in \(g^{-1}(\alpha_i)\) and the sets \(F_i\) are disjoint. Let \(I^*\) be an interval such that \(I^* \subset I_{k+1}\) and \(I^* \cap \bigcup_{i=1}^k F_i = \emptyset\). Choose a perfect set \(F_{k+1} \subset I^*\) such that \(g(F_{k+1}) = \alpha_{k+1} \in A\), \(F_{k+1}\) is nowhere dense in \(g^{-1}(\alpha_{k+1})\). Then \(F_{k+1} \cap F_i = \emptyset\) for \(i \neq k + 1\). Clearly the sum \(F = \bigcup_{i=1}^\infty F_i\) satisfies the conditions of the Lemma. Then there is a homeomorphism \(\chi\) from \(I\) onto itself such that \(|\chi(F)| = |I|\). Denote \(H = I \setminus \chi(F), h = \chi^{-1}\) and \(\Gamma = \{(x,y): x \in H, y + g(h(x)) \in D\}\). Certainly \(\chi(F)\) is a set of the first category of Baire. Thus \(H\) is dense in \(I\).

If \(B\) is a subset of \(H \times R\) we shall denote its \(x\)-projection, i.e. the set \(\{x: \exists y(x,y) \in B\}\) by \(\text{dom} B\), and its \(y\)-projection, i.e. the set \(\{y: \exists x(x,y) \in B\}\) by \(\text{rng} B\). For any \(t \in H\) we shall denote \(B_t = \{(x,y): x = t, (x,y) \in B\}\) and for any \(s \in R\) we shall denote \(B_s = \{(x,y): y = s, (x,y) \in B\}\). Since \(\Gamma\) is the union of the graphs of all functions \(z_n - g(h(x))\) where \(z_n \in D\), it follows that \(\text{rng} \Gamma = R\).

We shall say that two points \(A(x',y')\) and \(B(x'',y'')\) of \(\Gamma\) are equivalent, \(A \sim B\), if there exists a finite sequence of points \(A_i(x_i,y_i) \in \Gamma, i = 1, 2, \ldots, n\), such that \(A = A_1, B = A_n\), and for each \(i\) we have either \(x_i = x_{i+1}\) or \(y_i = y_{i+1}\). Denote by \(\Gamma/\sim\) the family of all equivalence classes for this equivalence relation. It is clear that for each \(t \in H\), \(\text{rng} \Gamma_t\) contains a translation of \(D\). Then \(\text{rng} \Gamma_t\) is a dense set. It follows from the definition of
equivalence that if \( G \) is an equivalence class, then \( G \cap \Gamma_i \neq \emptyset \) implies \( G \supseteq \Gamma_i \). In the same way if \( G \cap \Gamma^s \neq \emptyset \) then \( G \supseteq \Gamma^s \).

Now we shall prove that all dom \( \Gamma^s \) are dense in \( H \). Since \( g \) is not constant on any interval then for any open interval \( I^* \subseteq I \) there exist points \( x' \) and \( x'' \) in \( I^* \) for which \( g(h(x')) \neq g(h(x'')) \). Since \( D \) is dense there exists an \( r \in D \) such that \( r - s \) lies between \( g(h(x')) \) and \( g(h(x'')) \). Since \( g \circ h \) is Darboux there exists an \( x \in I^* \) for which \( g(h(x)) = r - s \). If \( x \in H \) then \( (x, s) \in \Gamma \) and therefore \( x \in \text{dom} \Gamma^s \cap I^* \).

If \( x \notin H \) then \( x \in \chi(F) \). Thus \( r - s = g(h(x)) = g(x^{-1}(x)) = \alpha_i \in A \). The level set \( g^{-1}(\alpha_i) \) is a perfect set and \( h(F) \cap g^{-1}(\alpha_i) = h(F) \cap g^{-1}(\alpha_i) \). The set \( h(F) \) is nowhere dense in \( g^{-1}(\alpha_i) \). It follows that there exists a point \( x^* \in I^* \cap [g^{-1}(\alpha_i) \setminus h(F)] \). Thus \( x^* \in H \cap I^* \) and \( \alpha_i = g(h(x^*)) = r - s \). Hence \( (x^*, s) \in \Gamma \) and \( x^* \in \text{dom} \Gamma^s \cap I^* \). This shows that \( \text{dom} \Gamma^s \) is dense in \( H \).

Suppose \( A(x', y') \sim B(x'', y'') \). Then
\[
y'' - y' = \sum_{i=1}^{n-1} (y'_{i+1} - y_i) = \sum_{i'} (y'_{i'} - y_{i'})
\]
de \( 1 \leq i' \leq n \) and \( y'_{i' + 1} - y_{i'} \neq 0 \). This inequality implies \( x_{i'} = x'_{i' + 1} \). Then
\[
y'_{i' + 1} - y_{i'} = y'_{i' + 1} - g(h(x'_{i' + 1})) - (y_{i'} - g(h(x_{i'}))).
\]
So \( y'_{i' + 1} - y_{i'} \) is the difference of two members of \( D \). The set of all such differences is countable. The set of finite sums of these differences is countable too. Each difference \( y'' - y' \) is such a sum of this form. Therefore the set of all differences of ordinates of points belonging to an equivalence class \( G \) is countable. Thus \( \text{rng} G \) is countable. Let us enumerate members of \( \text{rng} G \) as \( \{y_m\} \) and all rational subintervals of \( I \) as \( \{J_n\} \). Now enumerate all pairs \( (y_m, J_n) \) as \( \{(y_m, J_n)\} \). Since \( \text{dom} \Gamma^y \) is dense in \( H \), there exists a point \( t_i \in J_n \cap \text{dom} \Gamma^y \). Let \( t_1, t_2, \ldots, t_k \) be chosen such that \( t_i \in J_n \cap \text{dom} \Gamma^y \) and \( t_i \neq t_j \) for \( 1 \leq i < j \leq k \). Take \( t_{k+1} \in J_{n+1} \cap \text{dom} \Gamma^{y^{n+1}} \). Each set \( \text{dom} \Gamma^y \) contains a dense set of members of the sequence \( \{t_n\} \). Set \( d(x) = y_m \) for \( x = t_i \). Then \( \{d(x) \cap x = t_n \in J_m \} = \text{dom} G \). Since \( \bigcup_{G \in \Gamma^y} \text{rng} G = R \) the function \( d \) defined on \( \bigcup_{G \in \Gamma^y} \{t_n\} \) maps each intersection \( H \cap J_m \) onto \( R \). If \( G' \neq G'' \) then \( \text{dom} G' \cap \text{dom} G'' = \emptyset \). Therefore there is no contradiction in our extension of \( d \) from the subset of \( \text{dom} G \) to the sum \( \bigcup \text{dom} G \).

At present \( d \) is defined on a subset of \( H \). This subset is dense in \( I \), therefore each next extension of \( d \) on \( I \) must be Darboux. Let us fix a point \( c \in D \). Set \( d(x) = c - g(h(x)) \) for each point \( x \) at which we have not defined \( d(x) \). Since \( |H| = 0, g(h(x)) + d(x) = c \) almost everywhere.

We still have to prove that \( d - (g \circ h) \) maps \( I \) onto \( D \). Let \( G^0 \) be the equivalence class containing \( \Gamma^0 \). Then \( D \in \text{rng} G^0 \). For each \( x = t_n \) the point \( (x, d(x)) \) belongs to \( \Gamma \). Thus \( d(x) + g(h(x)) \in D \). For each \( x \neq t_n \) we have
$d(x) = c - g(h(x))$ where $c \in D$ so that $d(x) + g(h(x)) \in D$. This completes the proof of Theorem.

REFERENCES


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