MEASURABLE, TAIL DISINTEGRATIONS OF THE HAAR INTEGRAL ARE PURELY FINITELY ADDITIVE

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Abstract. There are countably additive probability measures, \( P \), and sub-sigma fields, relative to which \( P \) admits no proper, measurable, conditional distributions, except, possibly, those which are purely finitely additive. The usual fair, coin-tossing probability measure and the tail sigma field illustrate this phenomenon. More generally, every measurable, disintegration of the Haar integral of any compact metrizable group, \( G \), relative to the partition, \( \Pi \), of \( G \) which consists of the left cosets of any dense denumerable subgroup \( S \) of \( G \), or what comes to the same thing, relative to the sigma field of Haar-measurable subsets of \( G \) which are invariant under right translation by \( S \), is purely finitely additive.

This note relates to \([1]\) and \([2]\), but is logically independent of these references.

Let \( S \) be a denumerable, dense subgroup of a compact, metrizable group, \( G \), and let \( M \) be the unique, \( G \)-invariant mean defined on the space \( C(G) \) of all real-valued continuous functions on \( G \). Let \( \Pi = G/S \) be the set of left cosets \( gS \) of \( S \).

**Theorem 1.** Every measurable, \( \Pi \)-disintegration of \( M \) is purely finitely additive.

To say that \( \sigma \) is a measurable \( \pi \)-disintegration of \( M \) means this. For all \( f \) in the domain of \( M \),

\[
Mf = \int \sigma_g(f^g) \, d(g)
\]

where: (a) \( f^g \) is the trace of \( f \) on \( gS \), that is, the restriction of \( f \) to \( gS \); (b) \( \sigma_g \) is a mean which is supported by \( gS \), so that \( \sigma_g(gS) \) equals 1, \( \sigma_g = \sigma_{g'} \) if \( gS = g'S \), and \( \sigma_g \) is defined for \( f^g \) for all \( f \) in the domain of \( M \), and (c) \( \sigma_g(f^g) \) is integrable with respect to Haar measure \( dg \).

If, except for a set of \( g \)'s of Haar measure zero, \( \sigma_g \) is purely finitely additive, \( \sigma \) is said to be purely finitely additive.

The proof of Theorem 1 consists of two steps, the first of which is the
exhibition of a purely finitely additive disintegration $\hat{\sigma}$ of $M$, and the second is a demonstration that for every measurable disintegration $\sigma$ of $M$, $\hat{\sigma}_g = \sigma_g$ for almost all $g$.

Since, for each $g$, $gS$ is dense in $G$, the map $f \to f^g$ is a 1-1 mapping of $C(G)$ onto the space of uniformly continuous functions defined on $gS$. Consequently,

$$\hat{\sigma}_g(f_g) = Mf$$

defines $\hat{\sigma}$ uniquely. It is straightforward to verify that $\hat{\sigma}$ is a measurable $\Pi$-disintegration of $M$, henceforth called elementary.

**Lemma 1.** The elementary $\Pi$-disintegration of $M$ is purely finitely additive.

**Proof.** Let $\epsilon > 0$, and let $s_1, s_2, \ldots$ be an enumeration of the elements of $S$. As the Tietze extension theorem implies, there is, for each $n$, an $f_n \in C(G)$, $0 < f_n < 1$, such that $f_n(s_i)$ is 0 and $f_n$ is less than 1 on an open set of Haar measure at most $\epsilon 2^{-n}$. Let $f'_n$ designate the infimum of $f_n$, $i = 1, \ldots, n$, and verify: $f'_n \in C(G)$; $f'_n > f'_{n+1}$; $f'_n$ converges to 0 on $S$; and $M(f'_n)$ exceeds $1 - \epsilon$. Consequently, the restriction to $S$ of the $f'_n$ converges monotonely down to 0 everywhere on $S$ and, for any $g \in S$, $\hat{\sigma}_g(f'_n)$ exceeds $1 - \epsilon$. This implies that, for $g \in S$, $\hat{\sigma}_g$ is purely finitely additive. By appropriately translating the sequence $f'_n$, one concludes that every $\hat{\sigma}_g$ is purely finitely additive. Q.E.D.

A function $\phi$ defined on $G$ is $S$-invariant if, for every $s \in S$, the right translate of $\phi$ by $s$ is identical with $\phi$. If, for every integrable, $S$-invariant $\phi$, there is a constant $c$ such that $\phi = c$ almost certainly, then $S$ acts ergodically.

**Lemma 2.** Every dense subset of $G$ acts ergodically.

Surely Lemma 2 is known, but since I know of no reference, I supply a proof.

**Proof of Lemma 2.** Let $\phi$ be integrable and $S$-invariant for the dense subset $S$ of $G$. Let $f \in C(G)$ approximate $\phi$ in the $L_1$ norm. As shown by von Neumann [3], [4], there is a finite sequence of elements of $G$, $g_1, \ldots, g_n$, such that the average, $F$, of the right translates $f_i$ of $f$ by the $g_i$ is uniformly close to a constant. Because $s_i \in S$ can be chosen arbitrarily close to $g_i$, it may be supposed that the $g_i$ themselves are in $S$. Of course, the corresponding right translates $\phi_i$ of $\phi$ approximate $f_i$ in the $L_1$ norm as well as $\phi$ approximates $f$. Consequently, the average, $\theta$, of the $\phi_i$ approximates $F$, the average of the $f_i$. But, since $\phi$ is $S$-invariant, $\theta$ is $\phi$. So $F$ approximates $\phi$ in $L_1$, which implies that, for every $\epsilon > 0$, there is a constant $c$ such that $|\phi - c| < \epsilon$ except on a set of measure less than $\epsilon$. This implies that a single constant $c$ satisfies this condition for all $\epsilon > 0$. Now invoke countable additivity to see that $\phi = c$ almost surely. Q.E.D.

**Lemma 3.** Let $\sigma$ be a measurable, $\Pi$-disintegration of $M$. Then $\sigma_g = \hat{\sigma}_g$ for almost all $g$. 
Proof. As implied by Lemma 2, for each \( f \in C(G) \), there is a constant, \( c(f) \), and a subset of \( G \) of measure 0, say \( N(f) \), such that, for all \( g \) in the complement of \( N(f) \),

\[
\sigma_g(f^g) = c(f).
\]

In view of (1), \( c(f) = Mf \). Because \( G \) is compact and metrizable, there is a countable subset \( D \) of \( C(G) \) which is dense in \( C(G) \) for the topology of uniform convergence. Let \( N^c \) be the complement of the union of the null sets \( N(f) \) for \( f \) in \( D \). Summarizing, for all \( g \in N^c \), for all \( f \in D \),

\[
\sigma_g(f^g) = Mf.
\]

As is easily verified, for each \( g \), the set of \( f \) for which (4) holds is closed in the uniform topology. Consequently, for \( g \in N^c \), (4) holds for all \( f \in C(G) \). Equivalently, for all \( g \in N^c \), \( \sigma_g = \delta_g \). Q.E.D.

In view of Lemmas 1 and 3, Theorem 1 is seen to hold.

Consider now the special case in which \( G \) is the product of a denumerable number of copies of the cyclic group of order two. The usual fair, coin-tossing, probability measure is identical with normalized Haar measure on \( G \). And the atoms of the tail sigma field are simply the cosets of the subgroup \( S \) consisting of those elements of \( G \) which have no more than a finite number of nonzero coordinates. Hence, if Theorem 1 is applied to this example, one obtains

**Corollary 1.** Let \( \Omega \) be the space of infinite sequences of zeroes and ones, \( \mathbb{F} \) the field of finite-dimensional, Borel subsets of \( \Omega \), \( P \) the fair-coin probability measure on \( \mathbb{F} \). Then every Lebesgue measurable, proper, conditional distribution of \( P \) given the tail sigma field is purely finitely additive.

A similar example is obtained by letting \( G \) be the unit interval, with addition taken mod 1, \( dg \) equal to Lebesgue measure, and \( S \subset G \), the set of rationals.

Possibly the elementary \( \Pi \)-disintegration of \( M \) can be extended so as to be a \( \Pi \)-disintegration of all integrable functions, but the contrary seems to me to be more likely. Indeed, it is not unlikely that every \( \Pi \)-disintegrable measure defined for all Borel subsets of \( G \) is orthogonal to Haar measure on \( G \). A related conjecture is that "measurable" can be deleted from the statement of Theorem 1.

**References**


