MEASURABLE, TAIL DISINTEGRATIONS OF THE HAAR INTEGRAL ARE PURELY FINITELY ADDITIVE

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Abstract. There are countably additive probability measures, $P$, and sub-sigma fields, relative to which $P$ admits no proper, measurable, conditional distributions, except, possibly, those which are purely finitely additive. The usual fair, coin-tossing probability measure and the tail sigma field illustrate this phenomenon. More generally, every measurable, disintegration of the Haar integral of any compact metrizable group, $G$, relative to the partition, $\Pi$, of $G$ which consists of the left cosets of any dense denumerable subgroup $S$ of $G$, or what comes to the same thing, relative to the sigma field of Haar-measurable subsets of $G$ which are invariant under right translation by $S$, is purely finitely additive.

This note relates to [1] and [2], but is logically independent of these references.

Let $S$ be a denumerable, dense subgroup of a compact, metrizable group, $G$, and let $M$ be the unique, $G$-invariant mean defined on the space $C(G)$ of all real-valued continuous functions on $G$. Let $\Pi = G/S$ be the set of left cosets $gS$ of $S$.

**Theorem 1.** Every measurable, $\Pi$-disintegration of $M$ is purely finitely additive.

To say that $\sigma$ is a measurable $\pi$-disintegration of $M$ means this. For all $f$ in the domain of $M$,

\[ Mf = \int \sigma_g(f^g) \, d(g) \]

where: (a) $f^g$ is the trace of $f$ on $gS$, that is, the restriction of $f$ to $gS$; (b) $\sigma_g$ is a mean which is supported by $gS$, so that $\sigma_g(gS) = 1$, $\sigma_g = \sigma_g'$ if $gS = g'S$, and $\sigma_g$ is defined for $f^g$ for all $f$ in the domain of $M$, and (c) $\sigma_g(f^g)$ is integrable with respect to Haar measure $dg$.

If, except for a set of $g$'s of Haar measure zero, $\sigma_g$ is purely finitely additive, $\sigma$ is said to be purely finitely additive.

The proof of Theorem 1 consists of two steps, the first of which is the...
exhibition of a purely finitely additive disintegration \( \hat{\sigma} \) of \( M \), and the second is a demonstration that for every measurable disintegration \( \sigma \) of \( M \), \( \hat{\sigma}_g = \sigma_g \) for almost all \( g \).

Since, for each \( g \), \( gS \) is dense in \( G \), the map \( f \to f^g \) is a 1-1 mapping of \( C(G) \) onto the space of uniformly continuous functions defined on \( gS \). Consequently,

\[
\hat{\sigma}_g(f^g) = Mf
\]
defines \( \hat{\sigma} \) uniquely. It is straightforward to verify that \( \hat{\sigma} \) is a measurable \( \Pi \)-disintegration of \( M \), henceforth called elementary.

**Lemma 1.** The elementary \( \Pi \)-disintegration of \( M \) is purely finitely additive.

**Proof.** Let \( \varepsilon > 0 \), and let \( s_1, s_2, \ldots \) be an enumeration of the elements of \( S \). As the Tietze extension theorem implies, there is, for each \( n \), an \( f_n \in C(G), 0 < f_n < 1 \), such that \( f_n(s_i) = 0 \) and \( f_n \) is less than 1 on an open set of Haar measure at most \( \varepsilon 2^{-n} \). Let \( f'_n \) designate the infimum of \( f_n \), \( i = 1, \ldots, n \), and verify: \( f'_n \in C(G) \); \( f'_n > f'_{n+1} \); \( f'_n \) converges to 0 on \( S \); and \( M(f'_n) \) exceeds \( 1 - \varepsilon \). Consequently, the restriction to \( S \) of the \( f'_n \) converges monotonely down to 0 everywhere on \( S \) and, for any \( g \in S \), \( \hat{\sigma}_g(f'_n) \) exceeds \( 1 - \varepsilon \). This implies that, for \( g \in S \), \( \hat{\sigma}_g \) is purely finitely additive. By appropriately translating the sequence \( f'_n \), one concludes that every \( \hat{\sigma}_g \) is purely finitely additive. Q.E.D.

A function \( \phi \) defined on \( G \) is \( S \)-invariant if, for every \( s \in S \), the right translate of \( \phi \) by \( s \) is identical with \( \phi \). If, for every integrable, \( S \)-invariant \( \phi \), there is a constant \( c \) such that \( \phi = c \) almost certainly, then \( S \) acts ergodically.

**Lemma 2.** Every dense subset of \( G \) acts ergodically.

Surely Lemma 2 is known, but since I know of no reference, I supply a proof.

**Proof of Lemma 2.** Let \( \phi \) be integrable and \( S \)-invariant for the dense subset \( S \) of \( G \). Let \( f \in C(G) \) approximate \( \phi \) in the \( L_1 \) norm. As shown by von Neumann [3], [4], there is a finite sequence of elements of \( G, g_1, \ldots, g_n \), such that the average, \( F \), of the right translates \( f \) of \( f \) by the \( g \) is uniformly close to a constant. Because \( s_i \in S \) can be chosen arbitrarily close to \( g \), it may be supposed that the \( g \) themselves are in \( S \). Of course, the corresponding right translates \( \phi_i \) of \( \phi \) approximate \( f \) in the \( L_1 \) norm as well as \( \phi \) approximates \( f \). Consequently, the average, \( \theta_i \), of the \( \phi_i \) approximates \( F \), the average of the \( f_i \). But, since \( \phi \) is \( S \)-invariant, \( \theta \) is \( \phi \). So \( F \) approximates \( \phi \) in \( L_1 \), which implies that, for every \( \varepsilon > 0 \), there is a constant \( c \) such that \( |\phi - c| < \varepsilon \) except on a set of measure less than \( \varepsilon \). This implies that a single constant \( c \) satisfies this condition for all \( \varepsilon > 0 \). Now invoke countable additivity to see that \( \phi = c \) almost surely. Q.E.D.

**Lemma 3.** Let \( \sigma \) be a measurable, \( \Pi \)-disintegration of \( M \). Then \( \sigma_g = \hat{\sigma}_g \) for almost all \( g \).
Proof. As implied by Lemma 2, for each $f \in C(G)$, there is a constant, $c(f)$, and a subset of $G$ of measure 0, say $N(f)$, such that, for all $g$ in the complement of $N(f)$,

$$
\sigma_g(f^g) = c(f).
$$

In view of (1), $c(f) = Mf$. Because $G$ is compact and metrizable, there is a countable subset $D$ of $C(G)$ which is dense in $C(G)$ for the topology of uniform convergence. Let $N^c$ be the complement of the union of the null sets $N(f)$ for $f$ in $D$. Summarizing, for all $g \in N^c$, for all $f \in D$,

$$
\sigma_g(f^g) = Mf.
$$

As is easily verified, for each $g$, the set of $f$ for which (4) holds is closed in the uniform topology. Consequently, for $g \in N^c$, (4) holds for all $f \in C(G)$. Equivalently, for all $g \in N^c$, $\sigma_g = \delta_g$. Q.E.D.

In view of Lemmas 1 and 3, Theorem 1 is seen to hold.

Consider now the special case in which $G$ is the product of a denumerable number of copies of the cyclic group of order two. The usual fair, coin-tossing, probability measure is identical with normalized Haar measure on $G$. And the atoms of the tail sigma field are simply the cosets of the subgroup $S$ consisting of those elements of $G$ which have no more than a finite number of nonzero coordinates. Hence, if Theorem 1 is applied to this example, one obtains

**Corollary 1.** Let $\Omega$ be the space of infinite sequences of zeroes and ones, $\mathbb{F}$ the field of finite-dimensional, Borel subsets of $\Omega$, $P$ the fair-coin probability measure on $\mathbb{F}$. Then every Lebesgue measurable, proper, conditional distribution of $P$ given the tail sigma field is purely finitely additive.

A similar example is obtained by letting $G$ be the unit interval, with addition taken mod 1, $dg$ equal to Lebesgue measure, and $S \subset G$, the set of rationals.

Possibly the elementary $\Pi$-disintegration of $M$ can be extended so as to be a $\Pi$-disintegration of all integrable functions, but the contrary seems to me to be more likely. Indeed, it is not unlikely that every $\Pi$-disintegrable measure defined for all Borel subsets of $G$ is orthogonal to Haar measure on $G$. A related conjecture is that "measurable" can be deleted from the statement of Theorem 1.

**References**


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