ON CURVILINEAR CLUSTER SETS ON OPEN Riemann Surfaces

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Abstract. Every boundary point of the Kerékjártó-Stoïlow compactification of an open Riemann surface is the limit of a Jordan arc with this property: for every nonempty continuum in the extended complex plane there is a holomorphic function on the surface having the continuum as its cluster set along the arc.

Let \( R \) be an open Riemann surface and \( \{R_n\}_{n=1}^{\infty} \) a regular exhaustion of \( R \). Let \( R^* \) denote the Kerékjártó-Stoïlow compactification of \( R \) and \( \{G_n\}_{n=1}^{\infty} \) a determinant sequence of \( e \in \Delta \) satisfying \( \partial G_n \subset \partial R_n \), where \( \Delta = R^* - R \) and \( \partial X \) means the relative boundary of \( X \subset R \) with respect to \( R \).

The approximation theorem of Bishop (cf. [2]), which is applied in the proof of our Theorem, is stated as follows:

Let \( R' \) be an open Riemann surface and \( K' \) a compact subset with the property that no nonempty component of \( R' - K' \) is relatively compact. Let \( g' \) be a continuous function on \( K' \) which is holomorphic at interior points. Then for any \( \epsilon' > 0 \) there exists a holomorphic function \( f' \) on \( R' \) for which \( |f' - g'| < \epsilon' \) on \( K' \).

In this paper, we shall show the following:

Theorem. For each \( e \in \Delta \) and any nonempty continuum \( K \) in the Riemann sphere \( S \), there exist a holomorphic function \( f \) on \( R \) and a Jordan arc \( \gamma \) in \( R \) converging to \( e \) such that \( C_\gamma(f,e) = K \), where \( C_\gamma(f,e) \) denotes the cluster set of \( f \) on \( \gamma \).

Proof. Fix the chordal metric \( d \) on \( S \). Let \( U_n \) be the \( 1/n \)-neighborhood of \( K \) relative to \( d \). Evidently \( U_n \) is both open and connected. Let \( 0 = t_0 < t_1 < \cdots < t_n < \cdots, t_n \to 1 \), and for \( n \geq 1 \) let \( g_n: [t_{n-1}, t_n] \to U_n - \{\infty\} \) be continuous such that

1. no point of \( K \) is at a \( d \)-distance of more than \( 1/n \) from the image of \( g_n \),
2. for \( n \geq 2 \), \( g_n(t_{n-1}) = g_{n-1}(t_{n-1}) \).

Then \( \bigcup g_n \) defines a continuous \( g: [0, 1) \to S \) such that \( \bigcap_{1 \leq k < 1} g([t, 1])) = K \), where the bar implies closure.

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It is easy to see that there is an arc \( \gamma \) defined by a topological \( h: [0,1) \rightarrow R \) such that \( h(t) \rightarrow e \) as \( t \rightarrow 1, h([t_{n-1}, t_n]) \subset \overline{R}_{n+1} - R_n \), and \( \gamma \cap \partial G_n = \{h(t_{n-1})\} \). Define \( \varphi = g \circ h^{-1} \) on \( \gamma \subset R \) and extend \( \varphi \) by Tietze’s theorem to a continuous complex-valued function \( \varphi \) on \( R \). This is possible since \( g \) is never \( \infty \).

We may assume, without loss of generality, that \( \varphi = 0 \) on \( \overline{R}_1 \). Set \( \varphi_1 = 0 \) on \( R \) and \( \psi_2 = \varphi - \varphi_1 \) on \( \overline{R}_2 \); then from the approximation theorem of Bishop, there exists a holomorphic function \( f_2 \) on \( R \) for which \( |f_2 - \psi_2| < 2^{-3} \) on \( \overline{R}_1 \cup h([0, t_1]) \).

Next let \( \lambda(t) \) \( (t_1 \leq t < 1) \) be a continuous function with the property that \( \lambda(t_1) = \psi_2 \circ h(t_1), \lambda(t_2) = f_2 \circ h(t_2), |f_2 \circ h - \lambda| < 2^{-2} \) on \( [t_1, t_2] \) and \( \lambda = f_2 \circ h \) on \([t_2, 1]\). Let \( \varphi_2 \) be a continuous function on \( R \) such that \( \varphi_2 = \psi_2 \) on \( \overline{R}_2 \) and \( \varphi_2 = \lambda \circ h^{-1} \) on \((R - R_2) \cap \gamma \). Then we see that \( |f_2 - \varphi_2| < 2^{-2} \) on \( \overline{R}_1 \cup \gamma, \varphi_2 = f_2 \) on \((R - R_3) \cap \gamma \) and \( \varphi_2 = \varphi - \varphi_1 \) on \( \overline{R}_2 \).

By mathematical induction, we have a sequence \( \{f_n\}_{n=2}^\infty \) of holomorphic functions on \( R \) and a sequence \( \{\varphi_n\}_{n=2}^\infty \) of continuous functions on \( R \) with the property that \( |f_n - \varphi_n| < 2^{-n} \) on \( \overline{R}_{n-1} \cup \gamma, \varphi_n = f_n \) on \((R - R_{n+1}) \cap \gamma \) and \( \varphi_n = \varphi - (\varphi_1 + \varphi_2 + \cdots + \varphi_{n-1}) \) on \( \overline{R}_n \). Since \( \varphi_n = 0 \) on \( \overline{R}_{n-1}, \sum_{n=2}^\infty \varphi_n \) converges on \( R \) and \( \varphi = \sum_{n=2}^\infty \varphi_n \) on \( R \). Since \( |f_n| < 2^{-n} \) on \( \overline{R}_{n-1} \), \( \sum_{n=2}^\infty f_n \) converges uniformly on every compact subset of \( R \), and hence \( f = \sum_{n=2}^\infty f_n \) is holomorphic on \( R \).

Now for any \( \varepsilon > 0 \), there exists an \( N = N(\varepsilon) \) such that \( \sum_{n=N}^\infty |f_n - \varphi_n| < \varepsilon \) on \( \gamma \). Further since \( \sum_{n=2}^{N-1} |f_n - \varphi_n| = 0 \) on \((R - R_N) \cap \gamma \), we have \( |f - \varphi| < \varepsilon \) on \((R - R_N) \cap \gamma \). Since \( f - \varphi \rightarrow 0 \) along \( \gamma \), we see that \( f(h(t)) - g(t) = f(h(t)) - \varphi(h(t)) \rightarrow 0 \) as \( t \rightarrow 1 \). Clearly, the cluster set of \( f \) along \( \gamma \) is the cluster set of \( g \) along \([0,1]\), which we have seen is \( K \).

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