ON THE AUTOMORPHISM GROUP OF
COMPACT MEASURE HYPERBOLIC MANIFOLDS AND
COMPLEX ANALYTIC BUNDLES WITH
COMPACT MEASURE HYPERBOLIC FIBRES

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ABSTRACT. We prove the automorphism groups of the compact measure
hyperbolic manifolds are discrete.

1. Introduction. The aim of this note is to study the open problem: Given a
compact measure hyperbolic manifold, is the automorphism group finite? (See
[1, Problem 11].) Although we cannot prove or disprove this conjecture, the
following theorem is quite easy to show.

THEOREM 1. Given a compact measure hyperbolic manifold in the sense of
Eisenman-Kobayashi, then its automorphism group is discrete.

It is known that a compact manifold of general type must be measure
hyperbolic, but the converse is an open problem. The finiteness of the
automorphism group of a compact manifold of general type is a well-known
fact. (See [2] for a differential geometric proof.) Our conjecture would be
settled if one could prove the automorphism group is compact. The compact-
ness would follow from the existence of an invariant metric that induces the
underlying topology; it is unknown even if we assume the manifold to be
projective algebraic.

As an application of above result, we can prove the following theorems,
which might be regarded as a generalization of recent results of H. L. Royden
[3].

THEOREM 2. (1) A holomorphic fibre bundle with simply connected base and
compact measure hyperbolic fibres is holomorphically trivial.

(2) A holomorphic fibre bundle with compact measure hyperbolic fibre is a flat
bundle.

We also obtain the following nontrivial fact as a consequence of Theorems
1 and 2. For some related results one can consult a recent survey article of
Kobayashi (Intrinsic distance, measure and geometric function theory, Bull.
Theorem 3. A holomorphic fibre bundle with compact measure hyperbolic fibres and measure hyperbolic base is also measure hyperbolic.

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2. Definitions and proof of Theorem 1. Let $M$ be a complex manifold, $B_n$ the unit ball in $C^n$, and $V_B$ the volume form of the Bergmann metric in $B_n$.

The Eisenman-Kobayashi measure is defined as follows.

With respect to the local coordinates around a fixed point $P \in M$, one can write $K_M = k_M dz_1 \wedge d\bar{z}_1 \cdots dz_n \wedge d\bar{z}_n$, where

$$k_M(P) = \inf \{|g(0)| \int_B (dz_1 \wedge d\bar{z}_1 \cdots dz_n \wedge d\bar{z}_n) = g(x) \cdot V_B, f \text{ belongs to } M(B_n), \text{ and } f(0) = P\}$$

$x \in B_n$.

Definition. $M$ is measure hyperbolic if $K_M$ is nonzero almost everywhere (i.e. outside the union of finitely many complex analytic varieties).

Remark. It is easy to see that the measure on $C \times F$, where $F$ is any complex manifold and $C$ is the complex line, is identically zero. The measure also possesses the important property that it is volume-decreasing under holomorphic mappings.

Proof of Theorem 1. Let us write $A = \text{Aut} (M, M)$. Then $A$ is a complex Lie group acting holomorphically on $M$ by a famous theorem of Bochner and Montgomery. Suppose $A$ is not discrete. Let $V$ be a nonzero element of the Lie algebra of $A$ and $\exp(tV)$ the complex one parameter group acting in the direction of $V$. Then by a theorem of Bochner and Montgomery again, $\exp(tV)$ induces a nontrivial holomorphic vector field on $M$.

Since $M$ is measure hyperbolic, $K_M$ is nonzero outside the union $S$ of finitely many complex analytic varieties. The vector field cannot vanish identically on $M - S$, so we can choose $m \in M$ in such a way that both the holomorphic field and $K_M$ are not zero in a neighborhood of $m$.

Let $D_{n-1}$ be an $(n-1)$-dimensional complex analytic disc embedded in $M$ with center at $m$, where $D_{n-1}$ is transversal locally to the orbit of $\exp(tV)$ near $m$. Then $D_{n-1} \times \exp(tV)$ is contained in $M \times A$ in a natural way and so induces a holomorphic mapping $F: D_{n-1} \times C \to M$, such that $dF \neq 0$ close to $m$. However, the Eisenman-Kobayashi measure is identically zero in $D_{n-1} \times C$. By the volume-decreasing property, $K_M$ must vanish in a small neighborhood of $F(m)$. This is an obvious contradiction of the assumption that $M$ is measure hyperbolic.

3. Proofs of Theorems 2 and 3. (A) Every holomorphic fibre bundle with base $B$ and fibre $M$ can be described as a cocycle on $B$ with values in $A = \text{Aut} (M, M)$: There exists an open covering $\{U_i\}$ of $B$ such that the
transition functions \( f_{ij} : U_i \cap U_j \to A \) obey the cyclic rule \((f_{ij}) \cdot (f_{jk}) = (f_{ik})\) on \( U_i \cap U_j \cap U_k \), for all \( i, j, k \). But \( A \) is discrete, and \( U_i \) can be chosen to be connected; hence \( f_{ij} \) is constant on \( U_i \cap U_j \). This is the standard definition of a flat bundle.

(B) It is a well-known fact that the flat bundles can be constructed from homomorphisms of the fundamental group of \( B \) into \( A = \text{Aut} (M, M) \) (see Gunning, Lectures of vector bundles over Riemann surfaces, Princeton Univ. Press). If \( B \) is simply connected, then the representation is trivial, so that the bundle is holomorphically equivalent to \( M \times B \).

(A) and (B) together prove Theorem 2.

(C) Theorem 3 follows from the following two facts:

1. Let \( B' \) be the universal covering space of \( B \). Then \( B' \) is measure hyperbolic iff \( B \) is measure hyperbolic.

2. The product of two measure hyperbolic manifolds is also measure hyperbolic.

One observes that \( M \times B' \) covers the bundle with fibre \( M \) and base \( B \); our theorem follows immediately from the two assertions above.

REFERENCES

1. S. Kobayashi, Some problems on intrinsic distances and measures (preprint, Problem 11).

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