MONOTONE AND OSCILLATORY SOLUTIONS
OF $y^{(n)} + py = 0$

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Abstract. Monotone and oscillatory behaviors of the solutions with the property that $y(x)/x^2 \to 0$ as $x \to \infty$ or $y(x)/x \to 0$ as $x \to \infty$ are discussed. For example, it is shown that every nonoscillatory solution $y$, such that $y(x)/x \to 0$ as $x \to \infty$, monotonically tends to zero as $x \to \infty$, provided $n$ is odd, $p \equiv 0$, and $\int_0^\infty x^{n-2}p(x)\,dx = \infty$.

The differential equation to be considered is

$$y^{(n)} + py = 0,$$

where $p$ is continuous and of one sign on an interval $[a, \infty)$. Since the behaviors of solutions of (E) depend on the parity of $n$ and the sign of $p$, it is natural to classify (E) into the following four cases:

(i) $n$ even, $p \geq 0$,

(ii) $n$ odd, $p \geq 0$,

(iii) $n$ even, $p \leq 0$,

(iv) $n$ odd, $p \leq 0$.

In the sequel, (E$_i$), for example, shall denote the equation (E) satisfying condition (i). Likewise, (E$_{ii}$), (E$_{iii}$), and (E$_{iv}$) denote equation (E) satisfying (ii), (iii), and (iv), respectively.

A nontrivial solution of (E) is said to be oscillatory on $[a, \infty)$ if it has an infinity of zeros on $[a, \infty)$; otherwise, it is said to be nonoscillatory on $[a, \infty)$. If every solution of (E) is nonoscillatory on $[a, \infty)$, (E) is said to be nonoscillatory on $[a, \infty)$. On the other hand, if (E) has an oscillatory solution, (E) is said to be oscillatory.

Oscillatory and nonoscillatory behaviors of (E) have been extensively studied by a number of authors [1]–[13]. In particular, Anan’eva and Balaganowski [1] proved the following statements: If $p > 0$ and $\int_a^\infty x^{n-2}p(x)\,dx = \infty$, then every solution of (E) is oscillatory if $n$ is even, while every nonoscillatory
solution of (E) with odd $n$ satisfies
\[ \lim_{x \to \infty} y^{(n-1)}(x) = \cdots = \lim_{x \to \infty} y'(x) = \lim_{x \to \infty} y(x) = 0, \]
where the signs of $y, y', \ldots, y^{(n)}$ are preserved for sufficiently large $x$ and successively alternating. Similar results were obtained by Kondrat’ev [4] under different conditions imposed on the coefficient $p$. Central to the proofs of these results is that any solution $y$ with $y(x) \to 0$ as $x \to \infty$ is oscillatory, provided equation (E) meets certain requirements.

It is our aim to improve and extend some aspects of the above results. For example, it will be shown that every nontrivial solution $y$ of (E$_1$) or (E$_{iv}$) such that $y(x)/x^2 \to 0$ as $x \to \infty$, either is oscillatory or satisfies a sequence of inequalities, which has interesting consequences. We shall also prove that every nonoscillatory solution $y$ of (E$_{ii}$) such that $y(x)/x \to 0$ as $x \to \infty$ tends to zero as $x \to \infty$ if $\int_a^\infty x^{n-1} p(x) \, dx = \infty$.

**Theorem 1.** Assume that $p \neq 0$ on $[a, \infty)$ for every $a \leq a$. If $y$ is a nontrivial solution of (E$_i$) or (E$_{iv}$) such that $y \geq 0$ and $y(x)/x^2 \to 0$ as $x \to \infty$, then
\[ y(x) \geq 0, \quad y'(x) > 0, \quad y''(x) < 0, \]
and $y^{(k)}(x) \to 0$ monotonically as $x \to \infty$, $k = 2, 3, \ldots, n - 1$.

**Proof.** Put $y_1 = y, y_2 = y', \ldots, y_n = y^{(n-1)}$ and let $b$ be an arbitrary point of $[a, \infty)$. Then $y$ satisfies the system
\[ y_1(x) = y_1(b) + \int_b^x y_2(t) \, dt, \]
\[ \ldots \]
\[ y_{n-1}(x) = y_{n-1}(b) + \int_b^x y_n(t) \, dt, \]
\[ y_n(x) = y_n(b) - \int_b^x p(t) y_1(t) \, dt. \]
Suppose $y = y_1$ is a solution of (E$_i$). Then $\int_b^x p(t) y_1(t) \, dt$ is a nondecreasing, nonnegative function of $x$ and clearly is positive on an interval $[c, \infty)$ for some $c > b$. We claim that $y_n(b) > 0$. To prove this, assume the contrary: $y_n(b) \leq 0$. Then, $y_n(x)$ is nonpositive, nonincreasing on $[b, \infty)$ and
\[ y_n(c) = y_n(b) - \int_b^c p(t) y_1(t) \, dt < 0, \]
that is, $y_n(x) \leq y_n(c) < 0, x \in [c, \infty)$. Consequently, $y_{n-1}(x) \to -\infty$ as $x \to \infty$, irrespective of $y_{n-1}(b)$. This in turn implies $y_{k-1}(x) \to -\infty$ as $x \to \infty$, and successively $y_k(x) \to -\infty$ as $x \to \infty$, regardless of the value $y_k(b), k = 1, 2, \ldots, n - 1$. In particular, $y_1(x) = y(x) \to -\infty$ as $x \to \infty$, contrary to the
hypothesis that $y \geq 0$ on $[a, \infty)$. This contradiction proves $y_n(b) > 0$. Since $b$ is arbitrary, we conclude that $y_n(x) > 0$, $x \in [a, \infty)$. It is now easy to see that $y_n(x) \to 0$ as $x \to \infty$ for $n > 2$. If this were not the case, there would exist a constant $K > 0$ such that $y_n(x) > K$, $x \in [c_1, \infty)$, for some $c_1 \geq a$. However, this implies that $y(x) > K_1 x^{n-1}$ on $[c_2, \infty)$ for some constants $K_1 > 0$ and $c_2 > c_1$, contradicting the asymptotic behavior $y(x)/x^2 \to 0$ as $x \to \infty$. Next, we shall prove that $y_{n-1}(x) < 0$ if $n > 2$. Evidently, $y_n(x)$ is a monotonically increasing function. If $y_{n-1}(b) > 0$, then $y_{n-1}(x) \geq 0$ on $[b, \infty)$, and there would exist constants $C > 0$ and $d > b$ such that $y_{n-1}(x) > C$, $x \in [d, \infty)$. However, for $n > 2$, this again leads to the contradiction that $y(x) > Cx^{n-2}$, $x \in [d_1, \infty)$, for some $d_1 > d$. Thus, $y_{n-1}(b) < 0$, and $y_{n-1}(x) < 0$ since $b$ is arbitrary. Moreover, we must have $y_{n-1}(x) \to 0$ as $x \to \infty$, for otherwise we would again be led to the contradiction that $y(x) \to -\infty$ as $x \to \infty$. In this way, we can successively establish the inequalities

$$y_n(x) > 0, \quad y_{n-1}(x) < 0, \ldots, y_4(x) > 0, \quad y_3(x) < 0, \quad x \in [a, \infty),$$

with the property that $y_k(x) \to 0$ as $x \to \infty$, $k = 3, 4, \ldots, n$. Continuing this process, we deduce $y_2(x) > 0$ and $y_1(x) \geq 0$, $x \in [a; \infty)$. This proves the theorem for (E$_i$). The proof for (E$_{iv}$) is similar; in this case, we first prove that $y_n(x) < 0$ and $y_n(x) \to 0$ as $x \to \infty$, and continue as in the case of (E$_i$).

In a somewhat similar fashion, we can prove

**Theorem 2.** Assume that $p \neq 0$ on $[a_1, \infty)$ for every $a_1 \geq a$. If $y$ is a nontrivial solution of (E$_{ii}$) or (E$_{iii}$) such that $y \geq 0$ on $[a, \infty)$ and $y(x)/x \to 0$ as $x \to \infty$, then

$$y(x) > 0, \quad y'(x) < 0, \quad y''(x) > 0, \ldots, (-1)^n y^{(n-1)}(x) < 0,$$

$$x \in [a, \infty),$$

and $y^{(k)}(x) \to 0$ monotonically as $x \to \infty$, $k = 1, 2, \ldots, n - 1$.

In order to characterize the behaviors of solutions, we may reformulate Theorem 1 as follows:

**Corollary 1.** Suppose $p \neq 0$ on $[a_1, \infty)$ for every $a_1 \geq a$. Let $y$ be a nontrivial solution of (E$_i$) or (E$_{iv}$) such that $y(x)/x^2 \to 0$ as $x \to \infty$. Then either

(a) $y$ is oscillatory on $[a, \infty)$, or else

(b) $y \geq 0$ on $[b, \infty)$, for some $b \geq a$, and $y [-y]$ satisfies the inequalities in (1) of Theorem 1. In particular, $y [-y]$ increases [decreases] monotonically on $[b, \infty)$.

If $y$ is a nontrivial solution of (E$_i$) or (E$_{iv}$) such that $y(x) \to 0$ as $x \to \infty$, it cannot satisfy the inequalities in (1) of Theorem 1. Thus, we conclude by Corollary 1 that $y$ is oscillatory.

**Theorem 3.** If $\int_a^\infty x^{n-1} p(x) dx = \infty [-\infty]$, every nonoscillatory solution of (E$_i$) [(E$_{iv}$)] is unbounded on $[a, \infty)$.
Proof. Assume the contrary. Suppose there exists a nontrivial solution \( y \) which is bounded and positive on \([b, \infty)\), for some \( b \geq a \). Since \( y \) increases monotonically by Theorem 1, there exist two positive constants \( M_1 \) and \( M_2 \) such that \( M_1 \leq y(x) \leq M_2 \), \( x \in [b, \infty) \). Multiplying (E) by \( x^{n-1} \) and integrating \( n \) times by parts, we get

\[
x^{n-1}y^{(n-1)} - (n-1)x^{n-2}y^{(n-2)} + (n-1)(n-2)x^{n-3}y^{(n-3)} - \cdots + (-1)^{n-1}(n-1)!y = -\int_{b}^{x} t^{n-1}p(t)y(t)\,dt + C,
\]

where \( C \) is a constant.

If \( y \) is a solution of (E), then \( n \) is even, \( p \geq 0 \), and we get from (3),

\[
x^{n-1}y^{(n-1)} - (n-1)x^{n-2}y^{(n-2)} + \cdots + [(n-1)(n-2)\cdots 2]xy' - (n-1)!M_2 < -M_1\int_{b}^{x} t^{n-1}p(t)\,dt + C, \quad x \in [b, \infty).
\]

On the other hand, since \( y^{(n-1)} > 0, y^{(n-2)} < 0, \ldots, y' > 0 \) by Theorem 1, the left-hand side of (4) cannot tend to \(-\infty\) as \( x \to \infty \), while the right-hand side tends to \(-\infty\) as \( x \to \infty \). Therefore, inequality (4) cannot hold throughout \([b, \infty)\). This incompatibility proves that the solution \( y \) must be unbounded on \([a, \infty)\).

If \( y \) is a solution of (E), then \( n \) is odd, \( p \leq 0 \), and we get from (3),

\[
x^{n-1}y^{(n-1)} - (n-1)x^{n-2}y^{(n-2)} + \cdots - [(n-1)(n-2)\cdots 2]xy' + (n-1)!M_2 > -M_1\int_{b}^{x} t^{n-1}p(t)\,dt + C, \quad x \in [b, \infty).
\]

In this case, \( y^{(n-1)} < 0, y^{(n-2)} > 0, \ldots, y' > 0 \), due to Theorem 1. For this reason, the left-hand side of (5) cannot approach \( \infty \) as \( x \to \infty \), while the right-hand side approaches \( \infty \) as \( x \to \infty \). Hence, inequality (5) cannot hold throughout \([b, \infty)\), and our assumption that \( y \) is bounded must be false.

Theorem 3 may be restated as follows: If (E) ([E]) has a bounded nonoscillatory solution on \([a, \infty)\), then \( \int_{a}^{\infty} x^{n-1}p(x)\,dx < \infty \) \( [> -\infty] \). However, the above inequality is known to guarantee the nonoscillation of (E) ([E]) on \([a, \infty)\) [4], [11]. Thus, we have the following result.

**Corollary 2.** If (E) ([E]) has a bounded nonoscillatory solution defined on \([a, \infty)\), then (E) ([E]) is nonoscillatory on \([a, \infty)\).

According to a result of Kondrat'ev [4], any solution \( y \) of \( y^{(n)} + py = 0, p \in C[a, \infty), p \geq 0 \), such that \( y(x) \to 0 \) as \( x \to \infty \) and \( y(b) = 0 \), for some \( b \in [a, \infty) \), is oscillatory on \([a, \infty)\). In this connection, we have the following corollary of Theorem 2.

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Corollary 3. Suppose $p \neq 0$ on $[a_1, \infty)$ for every $a_1 \geq a$. Let $y$ be a nontrivial solution of $(E_{i_{ii}})$ or $(E_{i_{iii}})$ such that $y(x)/x \to 0$ as $x \to \infty$. If $y$ violates any of the inequalities in (2) of Theorem 2, then it is oscillatory on $[a, \infty)$. In particular, if $y^{(k)}(b) = 0$ for some $k$, $0 \leq k \leq n - 1$, and some point $b \in [a, \infty)$, then $y$ is oscillatory on $[a, \infty)$.

It is well known that equation $(E_{i_{ii}})$ and $(E_{i_{iii}})$ have a nontrivial solution $y$ satisfying the inequalities

$$(-1)^j y^{(j)}(x) \geq 0, \quad j = 0, 1, \ldots, n - 1, \quad x \in [a, \infty),$$

which is due to Hartman and Wintner [2]. In a recent paper [9, Theorem 4], Read showed that any solution $z = z(x)$ of $(E_{i_{iii}})$ satisfying (6), tends to zero as $x \to \infty$ if and only if $\int_a^\infty t^{n-1} p(t) dt = -\infty$. We shall establish a similar result for $(E_{i_{ii}})$.

**Lemma.** Suppose $y$ is a nontrivial solution of $(E_{i_{ii}})$ which satisfies (6). Then $y(x) \to 0$ as $x \to \infty$ if $\int_a^\infty x^{n-1} p(x) \, dx = \infty$.

**Proof.** Suppose $y(x)$ does not approach 0 as $x \to \infty$. Then there exists a positive constant $M$ such that $y(x) > M, x \in [a, \infty)$. Using (3) with $b$ replaced by $a$, and recalling that $n$ is odd and $p \geq 0$, we deduce

$$x^{n-1} y^{(n-1)} - (n - 1)x^{n-2} y^{(n-2)} + \cdots$$

$$- [(n - 1)(n - 2) \cdots 2] x y' + (n - 1)! M$$

$$< - M \int_a^x t^{n-1} p(t) dt + C_1, \quad x \in [a, \infty),$$

where $C_1$ is a constant. Since the left-hand side of this inequality is positive on $[a, \infty)$ because of (6), $\int_a^\infty x^{n-1} p(t) dt$ cannot tend to $\infty$ as $x \to \infty$. This proves the Lemma.

In view of Theorem 2, Theorem 4 of Read [9], and the Lemma, we have

**Theorem 4.** If $\int_a^\infty x^{n-1} p(x) \, dx = \infty [-\infty], every nonoscillatory solution $y$ of $(E_{i_{ii}})$ $(E_{i_{iii}})$ such that $y(x)/x \to 0$ as $x \to \infty$ tends monotonically to zero as $x \to \infty$.

Furthermore, if $(E_{i_{ii}})$ $(E_{i_{iii}})$ has two linearly independent solutions $y_1$ and $y_2$ such that

$$\lim_{x \to \infty} y_1(x)/x = \lim_{x \to \infty} y_2(x)/x = 0,$$

then $(E_{i_{ii}})$ $(E_{i_{iii}})$ is oscillatory. If either $y_1$ or $y_2$ is oscillatory, there is nothing to prove. Otherwise, consider the solution defined by $u(x) = y_2(a) y_1(x) - y_1(a) y_2(x)$. Since $\lim_{x \to \infty} u(x)/x = 0$ and $u(a) = 0$, the solution $u$ is oscillatory by Corollary 3.

Every nonoscillatory solution $y = y(x)$ of $(E_i)$ $(E_{i_{ii}})$ tends to zero as $x \to \infty$, provided

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\[ \int_a^\infty x^{n-2} p(x) \, dx = \infty \]

or

\[ p(x) \preceq \left( \bar{\lambda} + \epsilon \right) x^n, \quad \epsilon > 0, \]

where \( \bar{\lambda} \) is defined to be the maximum of the local maxima of the function \( f(x) = -x(x-1)(x-2) \cdots (x-n+1) \) [1], [4]. Using this fact and Corollary 1, we can easily conclude that every solution of \((E_i)\) is oscillatory if (7) or (8) is satisfied [1], [4]. Similarly, we can show that \((E_{ii})\) has a fundamental system consisting of \( n \) oscillatory solutions, provided either (7) or (8) is satisfied. For example, let \( y_i \) be the solution defined by the initial conditions \( y_i'(0) = \delta_{ij}, \quad i, j = 1, 2, \ldots, n. \) Then \( \{ y_1, y_2, \ldots, y_n \} \) is such a system (Corollary 3). Since equation \((E_{ii})\) is known to have a nonoscillatory solution [2], it also has a fundamental system consisting of one nonoscillatory solution and \( n - 1 \) oscillatory solutions if (7) or (8) is fulfilled. Kondrat’ev obtained similar results for \((E_{iii})\) and \((E_{iv})\) [4].

References