CONTRACTIONS OF CONVEX SETS

ROBERT E. JAMISON

ABSTRACT. In this paper it is shown that, in a vector space over any ordered field, a noninfinitesimal contraction of a convex set K can be written as an intersection of translates of K.

A subset K of a vector space over a totally ordered field is called convex provided \( \lambda x + (1 - \lambda)y \) is in K whenever \( x \) and \( y \) are in K and \( \lambda \) is a scalar such that \( 0 \leq \lambda \leq 1 \). Recalling that any ordered field \( F \) has characteristic zero, and hence contains a copy of the rational numbers, we shall say that a positive element \( \mu \) in \( F \) is infinitesimal if \( \mu < r \) for all positive rational numbers \( r \). (For further discussion and examples of ordered fields, see [1, Chapter 13].)

In this note we shall prove the following intersection theorem:

Theorem. Suppose K is a convex set in a vector space over an ordered field and \( \mu \) is a positive scalar less than 1. If \( \mu \) is not infinitesimal, then, for some set \( T \) of vectors,

\[ \mu K = \bigcap \{ K + t : t \in T \} \]

It is easy to see that this result is plausible by considering either a square or a triangle in the plane, or in fact, any closed convex set. Difficulties arise, however, in the case of a convex set which includes only a portion of its boundary—say, the open unit disk together with the points of its circumference with rational x-coordinate.

Proof of the theorem. Let \( \mu \) be a noninfinitesimal positive scalar less than 1. Set \( P = \mu K \) and suppose \( q \) is a point not in \( P \). To prove the theorem, we must find a vector \( t \) such that \( P \subseteq K + t \) but \( q \notin K + t \). We distinguish two cases:

Case I. For all \( x \) in \( P \), we have \( q + \mu(x - q) \in P \). Let \( t = (1 - \mu^{-1})q \) so that

\[ K + t = \mu^{-1}P + (1 - \mu^{-1})q = q + \mu^{-1}(P - q) \]

The vectors in \( P - q \) are all nonzero since \( q \) is not in \( P \). Thus \( q \) is not in \( K + t \). But if \( x \) is in \( P \), then \( q + \mu(x - q) \in P \) by the case hypothesis, so that \( x \in q + \mu^{-1}(P - q) = K + t \). Thus \( P \subseteq K + t \).

Received by the editors August 5, 1975 and, in revised form, March 24, 1976.

AMS (MOS) subject classifications (1970). Primary 52A05; Secondary 15A03, 12J15.

Key words and phrases. Contraction, convex set, intersection theorem, ordered field, translation, vector space.
Case II. There is a point $p$ in $P$ such that $q + \mu(p - q) \notin P$. Note that if $0 < \lambda < \mu$, then the segment from $q + \lambda(p - q)$ to $q + (p - q)$ contains $q + \mu(p - q)$. Since $q + (p - q) = p \in P$ and $P$ is convex, the assumption on $p$ forces $q + \lambda(p - q) \notin P$.

Now since $\mu$ is not infinitesimal, the positive integral powers of $1 - \mu$ become ultimately smaller than any preassigned positive rational number and, hence, smaller than any preassigned positive noninfinitesimal. Hence, $(1 - \mu)^n < \mu$ for some sufficiently large positive integer $n$. By the preceding note, $q + (1 - \mu)^n(p - q)$ cannot belong to $P$.

Let $m$ be the smallest positive integer such that $q + (1 - \mu)m(p - q) \notin P$, and set $v = q + (1 - \mu)^m(p - q)$. (Here $v = p$ if $m = 1$.) Then $v$ belongs to $P$. If $x$ is any point in $P$, then $(1 - \mu)v + \mu x \in P$ since $P$ is convex. Whence

$$x \in \mu^{-1}(P - (1 - \mu)v) = K + (1 - \mu^{-1})v.$$ 

Thus letting $t = (1 - \mu^{-1})v$, we have $P \subseteq K + t$. But

$$\mu q + (1 - \mu)v = q + (1 - \mu)(v - q) = q + (1 - \mu)^m(p - q) \notin P.$$ 

Consequently, $q$ does not belong to

$$\mu^{-1}(P - (1 - \mu)v) = K + (1 - \mu^{-1})v = K + t.$$ 

This completes the proof of the theorem.

We conclude with a simple one-dimensional example to show that the result cannot be extended to include infinitesimal contractions. Let $F$ be an ordered field which contains a positive infinitesimal element $\delta$ [1, p. 70]. Take the convex set $K = \{x \in F: x \geq 0 \text{ and for some integer } n, x \leq n\}$. If $\lambda \in F$ such that $\delta K \subseteq K - \lambda$, then $0 \in K - \lambda$ since $0 \in \delta K$. Thus $\lambda \in K$, so $\lambda + 1$ must also be in $K$. Whence $1 \in K - \lambda$. Thus 1 belongs to every translate of $K$ containing $\delta K$, but 1 does not belong to $\delta K$ since $\delta$ is infinitesimal.

Acknowledgement. The author would like to acknowledge the referee's suggestions of refinements in the proof of the theorem.

Bibliography