APPLICATIONS OF GROUP ACTIONS ON FINITE COMPLEXES TO HILBERT CUBE MANIFOLDS

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Abstract. For any compact Hilbert cube manifold $M$ such that $\tilde{H}_s(M, \mathbb{Z}_p) = 0$, there exists an embedding $g$ of $M$ into the Hilbert cube $Q$ such that $g(M)$ is the fixed point set of a semifree periodic homeomorphism of $Q$ with period $p$. A counterexample is given to the conjecture that any two proper homotopic period $p$ homeomorphisms of a Hilbert cube manifold such that the homeomorphisms revolve trivially about a unique fixed point are equivalent. A counterexample is also given for the case where the fixed point set is empty.

I. Introduction. This paper continues the study of group actions on Hilbert cube ($Q$) manifolds which essentially began with the work of R. Y. T. Wong [12]. That this study has begun in earnest is shown by the list of questions in [1]. Another recent paper in this area is [7].

The first part of this paper shows that certain finite-dimensional results generalize to $Q$ manifolds without the obstruction theory necessary in the finite-dimensional case.

The second part of this paper demonstrates that Wong's classification [12] of certain periodic homeomorphisms of $Q$ with unique fixed point does not generalize to $Q$ manifolds. Thus the theory of such actions on $Q$ manifolds should be richer than the theory of such actions on $Q$.

II. Definitions and notation. $Q = \prod_{i=1}^{\infty} [-1, 1]$, the Hilbert cube. A Hilbert cube manifold is a paracompact Hausdorff space admitting a cover of open sets homeomorphic to open sets of $Q$.

$f \simeq g$ means $f$ is homotopic to $g$. $f$ is properly homotopic to $g$ if there is a proper homotopy between $f$ and $g$. (A map is proper if the inverse image of every compact set is compact.)

$X \simeq Y$ means $X$ is homeomorphic to $Y$. Simple homotopy theory will be used. See [6] for a general reference. $X \simeq_S Y$ means $X$ and $Y$ have the same simple homotopy type.

$\operatorname{proj} \lim \{X_i, C_i\}$ is the inverse limit of the spaces $X_i$ with bonding maps $C_i$.

In the following definitions, let $H$ be a homeomorphism of a space $X$ with...
period \( p \); that is, \( p \) is the smallest integer such that \( H^p = \text{id} \).

The fixed point set of \( H \) is \( \{ x : h(x) = x \} \).

\( H \) is semifree if for all \( x \) such that

\[
H(x) \neq x, \quad x, \quad H(x), \quad H^2(x), \ldots, \quad H^{p-1}(x)
\]

are \( p \) distinct points.

\( H \) revolves trivially about its fixed point set \( K \) if for every neighborhood \( U \) of \( K \), there is a contractible neighborhood \( V \subset U \) of \( K \) such that \( H(V) = V \).

\( X/(H) \) is the quotient of \( X \) by the group generated by \( H \), i.e., \( X/\sim \) where \( x \sim H^i(x), \quad 1 < i < p - 1, \) for all \( x \in X \).

\[
X/(H(x_0)) = X/\sim \quad \text{where} \quad x_0 \sim H^i(x_0), \quad 1 < i < p - 1.
\]

Two period \( p \) homeomorphisms \( f_1 \) and \( f_2 \) of the space are equivalent if there exists a homeomorphism \( g \) such that \( f_2 = gf_1g^{-1} \).

**III. \( Q \) manifolds as fixed point sets.**

**Theorem 1.** Let \( M \) be a compact \( Q \) manifold satisfying \( \tilde{H}_*(M, Z_p) = 0 \) (the reduced singular homology with coefficients in the integers modulo \( p \)). There is an embedding \( g: M \to Q \) and a semifree, periodic homeomorphism \( H: Q \to Q \) of period \( p \) such that \( M \) is exactly the fixed point set of \( H \).

**Proof.** Lowell Jones [9] proved that if \( K \) is a finite CW-complex satisfying \( \tilde{H}_*(K, Z_p) = 0 \), then there exists a finite contractible CW-complex \( X \) containing \( K \) as a subcomplex and a semifree, period \( p \) homeomorphism \( g: X \to X \) such that \( K \) is the fixed point set of \( g \).

Now T. A. Chapman's Triangulation Theorem for \( Q \) manifolds [5] states that if \( M \) is a compact \( Q \) manifold, then there exist \( K \), a finite simplicial (hence CW-) complex, and a homeomorphism \( f: M \to K \times Q \). So \( \tilde{H}_*(M, Z_p) = \tilde{H}_*(K, Z_p) \).

For this \( K \), let \( h \) and \( X \) be as in the conclusion to Jones' result. Let \( i: K \to X \) be the inclusion map. Let \( g = i \times \text{id}: K \times Q \to X \times Q \). Let \( H = h \times \text{id}: X \times Q \to X \times Q \). By a theorem of J. E. West [11], \( X \times Q \approx Q \). So \( g \) is the desired embedding and \( H \) is the desired semifree periodic homeomorphism of period \( p \). \( \square \)

**IV. Counterexamples.** R. Y. T. Wong [12] proved that if \( f, g \) are semifree period \( p \) homeomorphisms on \( Q \) with a unique fixed point and revolve trivially about the fixed point, then \( f \) and \( g \) are equivalent. In [1], it was asked: If \( f \) and \( g \) are semifree period \( p \) homeomorphisms of a \( Q \) manifold \( M \) such that (1) they have the same fixed point set \( K \) and it is a single point or empty, (2) both \( f \) and \( g \) revolve trivially at \( K \), and (3) \( f \) is properly homotopic to \( g \), then is \( f \) equivalent to \( g \)?

The answer to both cases, \( K = \emptyset \) and \( K \) a single point, is no. The counterexample for the case \( K = \emptyset \) is a straightforward construction using lens spaces [6, Chapter V]. The counterexample for the second case will take a bit more work.

**Counterexample 1.** Let \( S^3 = \Sigma_1 \ast \Sigma_2 = \) the join of two copies of
Let $f_1: S^3 \to S^3$ be defined by
\[
f_1(te^{ix}, (1 - t)e^{iy}) = (te^{ix + 2\pi/7}, (1 - t)e^{iy + 2\pi/7}).
\]
Let $f_2: S^3 \to S^3$ be defined by
\[
f_2(te^{ix}, (1 - t)e^{iy}) = (te^{ix + 4\pi/7}, (1 - t)e^{iy + 2\pi/7}).
\]
Let $M = S^3 \times Q$, be a $Q$ manifold.
Clearly $g$ can be defined such that $g: S^3 \times I \to S^3$, $g_0 = f_1$, $g_1 = f_2$. So $f_1 \simeq f_2$. Define $F_i: M \to M$ by $F_i = f_i \times \text{id}_Q$, $i = 1, 2$. Each $F_i$ is a semifree period 7 homeomorphism of $M$. Define $G: F_1 \simeq F_2$ by $G = g \times \text{id}_Q$. So conditions (1) and (3) of the question are met and (2) is vacuously satisfied. (The spaces involved are compact so all homotopies are proper.)
Suppose there exists $H$ such that $F_2 = HF_1H^{-1}$. Consider the following commutative diagram where $h$ is the naturally defined homeomorphism induced by $H$ and $q_i: M \to M/\{F_i\}$ is a naturally defined quotient map, $i = 1, 2$.

If $H$ existed, $M/\{F_1\} \simeq M/\{F_2\}$. But $M/\{F_i\} \simeq S^3/(f_i) \times Q$, $i = 1, 2$. By [6, §27], $S^3/(f_1) \simeq L(7, 1)$, $S^3/(f_2) \simeq L(7, 2)$, where $L(p, q)$ is the standard 3-dimensional lens space.

Chapman's Classification Theorem [5] states that if $X$ and $y$ are compact, connected $Q$ manifolds and if $X \simeq K \times Q$, and $Y \simeq L \times Q$ ($K$, $L$ finite simplicial complexes) are any two triangulations, then $X \simeq Y$ if and only if $K$ and $L$ have the same simple homotopy type. But $L(7, 1) \not\simeq L(7, 2)$ [6, §31] and so $M/\{F_1\} \not\simeq M/\{F_2\}$. So $H$ cannot exist and $f_1$ and $f_2$ are not equivalent.

**Counterexample 2.** Let $S^3$, $f_1$, $f_2$ be as in Counterexample 1. Let $T = S^3/(f_i(0, e^{\text{im}(0)})$. Note that $f_i(0, e^{\text{im}(0)}) = f_i(0, e^{\text{im}(0)})$ for all $i$. Let $q: S^3 \to T$ be the quotient map. From the definition of $T$, $f_1$, and $f_2$, $k_1 = qf_1q^{-1}$ and $k_2 = qf_2q^{-1}$ are well-defined semifree period 7 homeomorphisms of the CW-complex $T$.

Let $\ast$ denote the unique fixed point of the actions $k_1$ and $k_2$. Clearly $k_1$ and $k_2$ revolve trivially at $\ast$. Write $Q = \prod_{i=1}^{\infty}D_i$ where each $D_i$ is a two disk.
centered at the origin. Let \( r_i: D_i^2 \to D_i^2 \) be the standard rotation of \( D_i \) through \( \frac{2\pi}{7} \) radians. Define \( R: Q \to Q \) by \( R(x_1, x_2, \ldots) = (r_1(x_1), r_2(x_2), \ldots) \) where \( x_i \in D_i \) for all \( i \). By a theorem of West [11], \( T \times Q \) is a \( Q \) manifold.

Define \( K_i = k_i \times R: T \times Q \to T \times Q, i = 1, 2 \). Each \( K_i \) is a semifree period 7 homeomorphism revolving trivially around the same unique fixed point. \( J = g g^{-1} \times R \) is clearly a homotopy between \( K_1 \) and \( K_2 \) where \( g \) is the homotopy from the previous example.

Now suppose \( K_1 \) was equivalent to \( K_2 \), that is there exists \( P \) such that \( K_2 = P K_1 P^{-1} \). Then \( P \) would induce a homeomorphism \( \tilde{P} \) such that the following diagram would commute.

\[ \begin{array}{ccc}
T \times Q & \xrightarrow{S_1} & T \times Q/{K_1} \\
\downarrow P & & \downarrow \tilde{P} \\
T \times Q & \xrightarrow{S_2} & T \times Q/{K_2}
\end{array} \]

\( S_1 \) and \( S_2 \) are the naturally induced quotient maps. This would imply that \( T \times Q/{K_1} \cong T \times Q/{K_2} \). But it will now be shown that \( (T \times Q/{K_1}) \times Q \cong L(7, 1) \times Q \) and \( (T \times Q/{K_2}) \times Q \cong L(7, 2) \times Q \), and so \( P \) cannot exist.

Define

\[ T_0 = T/\{k_1\} \times (0, 0, \ldots) \times Q, \]

and

\[ T_n = \frac{T \times \prod_{i=1}^{n} D_i}{k_1 \times \prod_{i=1}^{n} r_i} \times (0_{n+1}, 0_{n+2}, \ldots) \times Q \]

for all \( n \geq 1 \).

Define \( C_n: T_{n+1} \to T_n \) by

\[ C_n = \text{id}_T \times \prod_{i=1}^{n} \text{id}_{D_i} \times C_{n+1} \times \text{id} \]

where \( C_{n+1} \) is induced by the radial collapse of \( D_{n+1} \) to the origin. By the previously mentioned theorem of West, \( T_n \) is a \( Q \) manifold. Each \( C_n \) is a \( CE \)-mapping and hence, by Chapman's \( CE \)-mapping Theorem [3], [4] (see [8] for a short proof), each \( C_i \) is a near homeomorphism. So by Brown's Inverse Limit Theorem [2], \( \text{proj lim} \{ T_i, C_i \} \cong T_0 \). It is easy to put a metric on \( T \times Q \) so that each \( C_i \) is within \( 2^{-i} \) of the identity map. Since each \( C_i \) is nonexpansive, the family \( \{ C_n \circ \cdots \circ C_{n+j} \} \cup_{j=0}^{\infty} \) is an uniformly equicontinuous family of maps. Certainly \( \cup_{i=1}^{\infty} T_i \times \right \) is dense in \( (T \times Q)/\{K_1\}) \times Q \) so by a lemma of R. Schori and D. Curtis [10], \( ((T \times Q)/\{K_1\}) \times Q \cong \)
proj lim\{T, C_i\}. So \(((T \times Q)/\{K_1\}) \times Q \cong T_0\). But \(T_0 = (T/\{k_1\}) \times (0, 0, \ldots) \times Q \cong L(7, 1) \times Q\). The proof that \(((T \times Q)/\{K_2\}) \times Q \cong L(7, 2) \times Q\) is obviously identical. So \(K_1\) and \(K_2\) are not equivalent. □

REFERENCES

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