A COUNTABLY COMPACT $k'$-SPACE NEED NOT BE COUNTABLY BI-$k$

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Abstract. An example is given of a countably compact $k'$-space that is not countably bi-$k$. Interest for this example arises from a recent paper of Michael, Olson, and Siwiec and from a 1972 paper of E. Michael, both of which discuss mapping characterizations of a range space. The construction of the example assumes the continuum hypothesis.

1. Introduction. In recent years Arhangel'skiï [1], [2], Siwiec [10], and Michael [6] characterized images of certain kinds of spaces under certain kinds of mappings. These characterizations are summarized in Table 1 of [6], part of which we now reproduce:

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
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</thead>
<tbody>
<tr>
<td>4</td>
<td>strongly $k'$</td>
<td>countable</td>
<td>countably</td>
<td>countably</td>
</tr>
<tr>
<td></td>
<td>base</td>
<td>bi-sequential</td>
<td>bi-$k$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$k'$</td>
<td>Fréchet $\mathcal{N}_0$</td>
<td>Fréchet</td>
<td>singly bi-$k$</td>
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All the terms in this table are defined in [6]. Those terms having a direct bearing on the example in this paper will be defined in §2. The two entries in each of Columns C and D coincide in the presence of countable compactness; for Column C this was proved in [5, (D) and (C), p. 983], and for Column D in [3, Corollary 3 to Theorem 7] and in [8, Theorem 5.1]. In this note, we show this is not the case in Columns B and E. (It seems rather plausible that this should be so, because [7, Diagrams 1.2 and 1.3, Proposition 2.4] imply that, under rather mild restrictions (e.g., $X$ Lindelöf, or $Y$ a Fréchet space, or every $y \in Y$ a $G_δ$), $Y$ countably compact and regular implies that every hereditarily quotient map $f: X \to Y$ is countably bi-quotient. Nevertheless, the answer for both Columns B and E is, in general, negative, as the following example shows. The construction of the example assumes the continuum hypothesis, which we indicate by [CH].

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1 The entries in Row 4 (respectively, Row 5) of [6, Table 1] are the images under countably bi-quotient (respectively, hereditarily quotient) maps of certain kinds of spaces. Those spaces are: (B) locally compact, paracompact spaces, (C) separable metrizable spaces, (D) metrizable spaces, and (E) paracompact $M$-spaces.

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Example 1.1 [CH]. There exists a completely regular, countably compact \( k' \)-space \( Y \) that is not countably bi-\( k \).

As the proof will show, \( Y \) actually has the following property, which is stronger than countable compactness: Every infinite subset of \( Y \) has an infinite subset with compact closure.

2. Some preliminaries. We begin with the appropriate definitions. A space \( Y \) is a \( k' \)-space [2, Chapter III, Definition 3.2] if whenever \( A \subset Y \) and \( y \in \overline{A} \) (the closure of \( A \)), then there exists a compact set \( K \subset Y \) such that \( y \in (A \cap K)^- \). A space \( Y \) is countably bi-\( k \) [6, Definition 4.E.1] if whenever \( (A_n) \) is a decreasing sequence of subsets with a common accumulation point \( y \), then there exists a \( k \)-sequence \( (K_n) \) such that \( y \in (A_n \cap K_n)^- \) for every \( n \). Here \( (K_n) \) is called a \( k \)-sequence if: (1) \( K_n \supseteq K_{n+1} \) for every \( n \), (2) \( K = \cap_n K_n \) is compact, and (3) if \( U \supseteq K \) and \( U \) is open, then \( U \supseteq K_n \) for some \( n \).

In the construction of Example 1.1, the Stone-Cech compactification \( \beta N \) of the set \( N \) of nonnegative integers (with the discrete topology) plays a central role, and we make the convention that, for subsets of \( \beta N \), “closure” means closure in \( \beta N \). Thus the “closure” of any such set is necessarily compact. Following [4, 6.S, pp. 98–99], \( A' \) shall denote the boundary in \( \beta N \) of a subset \( A \) of \( N \). We remark that \( N' = \beta N \setminus N \) is closed in \( \beta N \) and refer to [4, ibid.] for more details.

It is worthwhile to establish some lemmas to aid in the construction of the example, the first of these lemmas to be of a set-theoretic nature.

Lemma 2.1. Let \( \{W_a\}_{a<\Omega} \) be a strictly decreasing family of sets indexed by the first uncountable ordinal \( \Omega \), let \( W = \cap_a W_a \), and let \( \{G_a\}_{a<\Omega} \) be any family of sets satisfying \( G_a \cap W \neq \emptyset \) for all \( a \) and \( \beta \). Then there exists an increasing function \( \phi: [0, \Omega) \to [0, \Omega) \) such that \( G_a \cap W_{\phi(a)} \neq \emptyset \) for all \( a \).

Proof. Let \( \phi(0) = 0 \) and \( x_0 \in G_0 \cap W_{\phi(0)} \). Suppose inductively that \( x_\alpha \in G_\alpha \cap W_{\phi(\alpha)} \) for every \( \alpha < \beta < \Omega \), and whenever \( \alpha < \gamma < \beta \), we have \( \phi(\alpha) < \phi(\gamma) \) and \( x_\alpha \notin W_{\phi(\gamma)} \). Define

\[
\phi(\beta) = \sup\{\min\{\gamma: x_\alpha \notin W_\gamma\}: \alpha < \beta\}
\]

and let \( x_\beta \in G_\beta \cap W_{\phi(\beta)} \). Then \( \phi \) is as required.

Lemma 2.2. If \( A \) and \( B \) are \( F_\alpha \) subsets of \( \beta N \), and if \( \overline{A} \cap B = A \cap \overline{B} = \emptyset \), then \( \overline{A} \cap \overline{B} = \emptyset \).

Proof. Since \( \beta N \) is normal, an easy induction shows that \( A \) and \( B \) can be separated by disjoint open sets \( U \) and \( V \). Since \( \beta N \) is extremally disconnected (see [4, 6.M, p. 96]), \( U \cap \overline{V} = \emptyset \).

We complete the preliminaries with the following technical lemma.

Lemma 2.3 [CH]. Suppose \( S \) is an open \( F_\alpha \) in \( N' \). Then there exist a decreasing family \( \{V_a\}_{a<\alpha} \) of open-closed subsets of \( N' \), and a subset \( P \) of \( N' \setminus \overline{S} \) satisfying the following conditions:
(a) $\{V_a\}_{a<\Omega}$ is a base for the neighborhoods of $\overline{S}$ in $N'$.
(b) $P \setminus V_a$ is countable for all $a$.
(c) $P$ is relatively discrete and, hence, nowhere dense in $N'$.
(d) If $A \subset N'$, if $A \setminus V_a$ is countable for all $a$, and if $A \cap P \subset \overline{S}$, then $\overline{A} \cap P \subset \overline{S}$.
(e) If $F$ is open-closed in $N'$, and if $(F \setminus \overline{S})^c \cap \overline{S} \neq \emptyset$, then $F \cap P \neq \emptyset$.

**Proof.** Let $\mathcal{Q}$ be the collection of all open-closed sets in $N'$ that contain $S$. Write $\mathcal{Q} = \{U_a : a < \Omega\}$, and let $B_a = N' \setminus U_a$. For each $\beta < \Omega$, $S$ and $\bigcup \{B_a : a < \beta\}$ are disjoint open $F_\beta$'s and hence have disjoint closures. Since these closures are compact, there exists an open-closed set in $N'$ containing $S$ and disjoint from $\bigcup \{B_a : a < \beta\}$. That is, $\bigcap \{U_a : a < \beta\}$ contains some $U_\gamma$. Let $W_0 = U_0$ and let $W_\beta$ be the first $U_\gamma$ contained in $U_\beta \cap (\bigcap \{W_a : a < \beta\})$. Then $\{W_a\}_{a<\Omega}$ is a decreasing family of open-closed subsets of $N'$ and a base for the neighborhoods of $\overline{S} = \bigcap \{V_a : a < \Omega\}$, since each $W_a$ is compact.

Let $\mathcal{S}$ be the collection of all open-closed sets $F$ in $N'$ such that $(F \setminus \overline{S})^c \cap \overline{S} \neq \emptyset$. Write $\mathcal{S} = \{F_a : a < \Omega\}$. Since $F_a \setminus \overline{S}$ is open in $N'$, the set $G_a$ of $P$-points in $F_a \setminus \overline{S}$ is dense in $F_a \setminus \overline{S}$ (see [4, 6.V, p. 100]). Thus $G_a \cap \overline{S} \neq \emptyset$ and hence $G_a \cap W_\beta \neq \emptyset$ for all $a$ and $\beta$. By Lemma 2.1, there exists an increasing function $\phi : [0, \Omega) \to [0, \Omega)$ such that for all $a$, $G_a \cap W_{\phi(a)} \setminus W_{\phi(a+1)} \neq \emptyset$. Let $V_a = W_{\phi(a)}$, and let $P$ be the set obtained by choosing one $P$-point from each $F_a \cap V_{a+1}$. Thus (a), (b), and (e) are clearly satisfied. Since the sets $V_a \setminus V_{a+1}$ are open and pairwise disjoint, $P$ is relatively discrete.

Since $P$ is relatively discrete, $P$ is locally compact and, hence, open in $\overline{P}$. If the interior in $N'$ of $\overline{P}$ were nonempty, then its intersection with $P$ would be a nonempty open subset of $N'$, in which case $N'$ would have isolated points (again because $P$ is relatively discrete), which it does not. Thus $P$ is nowhere dense, and (c) is satisfied.

For (d), suppose $A \subset N'$, $A \setminus V_a$ is countable for all $a$, and $A \cap P \subset \overline{S}$. If $x \in (\overline{A} \cap P) \setminus \overline{S}$, then $x \notin V_a$ for some $a < \Omega$. Then $x$ belongs to the closures of the two countable sets $A \setminus V_a$ and $P \setminus V_a$. Since a $P$-point cannot be an accumulation point of a countable set in $N'$, $(A \setminus V_a)^c \cap P = \emptyset$. Since $A \cap P \subset \overline{S}$, $\overline{P} \cap A \setminus V_a = \emptyset$. By Lemma 2.2, $(A \setminus V_a)^c \cap P \setminus (A \setminus V_a)^c = \emptyset$, contradicting the fact that $x$ is in this intersection. Hence $(\overline{A} \cap P) \setminus \overline{S} = \emptyset$, completing the proof of the lemma.

3. The construction and proof of the example. Partition $N$ into an infinite collection of infinite subsets: $N = \bigcup_{n=1}^{\infty} S_n$, each $S_n$ infinite, and $S_n \cap S_m = \emptyset$ if $n \neq m$. Let $S = \bigcup_{n=1}^{\infty} S_n'$. Then $S$ is an open $F_\delta$ in $N'$. Assuming [CH], let $\{V_a\}_{a<\Omega}$ and $P$ be as in Lemma 2.3.

Define $X = \beta N \setminus \overline{P}$, define $Y = X/\overline{S}$ ($\overline{S}$ is identified to a point), and let $f$ denote the (necessarily perfect) identification map $f : X \to Y$. Let $s$ denote $f(\overline{s})$, considered as an element of $Y$. Observe $Y \setminus \{s\}$ is homeomorphic to $\beta N \setminus (P \cup S)$, and hence every point of $Y \setminus \{s\}$ has an open-closed, compact neighborhood.
First, we show \( X \), and hence \( Y \), is countably compact. Suppose \( A \subseteq X \) is countably infinite. We shall produce an infinite subset \( B \) of \( A \) with \( \overline{B} \) (necessarily compact) a subset of \( X \).

Case 1. \( A \cap N \) is infinite: Since \( (A \cap N)' \) is open in \( N' \) and \( P \) is nowhere dense in \( N' \), \( (A \cap N)' \setminus P \) is nonempty and open in \( N' \). Hence there is an infinite subset \( B \) of \( A \cap N \) such that \( \overline{B} \cap \overline{P} = \emptyset \). Thus \( \overline{B} \) is a compact subset of \( X \).

Case 2. \( A \cap N' \) is infinite: In this case, let \( B = A \cap N' \). Then \( \overline{B} \subseteq X \) by Lemma 2.3(d).

Next, \( Y \) is a \( k' \)-space. Let \( A \subseteq Y \) and \( y \) an accumulation point of \( A \). We must produce a compact set \( K \subseteq Y \) such that \( y \) is an accumulation point of \( A \cap K \). If \( y \neq s \), then \( y \) has a compact neighborhood and there is nothing to prove. We assume \( y = s \) and \( y \notin A \). Then \( f^{-1}(A)' \cap \overline{S} \neq \emptyset \).

Case 1'. \( (f^{-1}(A) \cap N)' \cap \overline{S} \neq \emptyset \): Then \( (f^{-1}(A) \cap N)' \) intersects \( S_n' \) for some \( n \), so \( f^{-1}(A) \cap S_n \) is infinite for this \( n \), in which case \( K = f(S_n') \) is compact and \( y \in (A \cap K)' \).

Case 2'. \( (f^{-1}(A) \cap N')' \cap \overline{S} \neq \emptyset \): Let \( x_\alpha \in f^{-1}(A) \cap V_\alpha \) for each \( \alpha < \Omega \). Then \( \{x_\alpha : \alpha < \Omega\} \setminus P_\beta \) is countable for all \( \beta \), and \( \{x_\alpha : \alpha < \Omega\} \setminus P = \emptyset \), so by Lemma 2.3(d), \( \{x_\alpha : \alpha < \Omega\}' \subseteq X \). Then \( K = f(\{x_\alpha : \alpha < \Omega\}') \) as required.

Finally, \( Y \) is not countably bi-\( k \). Let \( A_n = \bigcup_{k>n} f(S_k) \). Then \( s \in \overline{A_n} \) for all \( n \). Suppose \( (K_n) \) is a \( k \)-sequence with \( s \in \bigcap_n K_n = K \), where \( K \) is compact and \( s \in (A_n \cap K_n)' \) for all \( n \). Let \( B_n = f^{-1}(A_n \cap K_n) \). Then \( B_n \cap \overline{S} \neq \emptyset \) for all \( n \).

We claim \( B_n' \cap P \neq \emptyset \) for all \( n \). By Lemma 2.3(e), it suffices to show \( (B_n' \cap \overline{S})' \cap \overline{S} \neq \emptyset \) for each \( n \). If this intersection were empty for some \( n \), then for some \( \alpha \), \( (B_n' \cap \overline{S}) \cap V_\alpha = \emptyset \), so \( B_n' \cap \overline{V_\alpha} \) is a compact subset of \( \overline{S} \). Since \( B_n' \cap \overline{V_\alpha} \) is open in \( N' \), and \( N' \setminus S \) is a \( G_\delta \), \( B_n' \cap \overline{V_\alpha} \cap N' \setminus S \) is a \( G_\delta \) in \( N' \) that is contained in \( S \setminus S \), which has void interior in \( N' \). Therefore \( B_n' \cap \overline{V_\alpha} \cap N' \setminus S = \emptyset \). That is, \( B_n' \cap \overline{V_\alpha} \subseteq S = \bigcup_k S_k' \), and since we have an open (in \( N' \)) cover of a compact set, \( B_n' \cap \overline{V_\alpha} \subseteq \bigcup_{k<m} S_k' \) for some \( m \). Then

\[ B_{n+m}' \cap \overline{V_\alpha} \subseteq \bigcup_{k<m} S_k' \cap (f^{-1}(A_{n+m})')' = \emptyset, \]

from which it follows that \( B_{n+m}' \cap \overline{S} = \emptyset \), a contradiction.

Let \( Q \) be the set obtained by choosing one point from each \( B_n' \cap P \). Then \( \overline{Q} \subseteq \overline{P} \setminus \overline{S} \) by Lemma 2.2. Hence, \( \overline{Q} \cap X = \emptyset \), so \( \overline{Q} \cap f^{-1}(K) = \emptyset \). Separating these two compact subsets of \( \beta N \), there exists an open-closed set \( U \subseteq \beta N \) such that \( \overline{Q} \subseteq U \) and \( f^{-1}(K) \cap U = \emptyset \). Then \( f(U \cap X) = \emptyset \) is closed in \( Y \), disjoint from \( K \), but intersects every \( K_n \), since \( Q \subseteq U \) implies \( U \cap B_n \neq \emptyset \) and, hence, \( f(U \cap X) \cap (A_n \cap K_n) \neq \emptyset \). That contradicts \( (K_n) \) being a \( k \)-sequence and completes the proof.

4. Concluding comment. We conclude with a comment about the other entries in Column E of [6, Table 1]. Those other entries are: (1) paracompact
$M$-spaces, (2) spaces of pointwise countable type, (3) bi-$k$-spaces, and (6) $k$-spaces. We do not know whether there is an example of a countably compact bi-$k$-space that is not of pointwise countable type. Otherwise, none of the entries in Column E of [6, Table 1] coincide in the presence of countable compactness. If $Y$ is the space of Example 1.1, then $Y \times I$ is a countably compact $k$-space that is not singly bi-$k$ (using [6, Proposition 4.E.4]). Arhangel'skiǐ [3, p. 1187] has an interesting example of a sequentially compact countably bi-$k$-space that is not bi-$k$, and the ordinal space $[0, \Omega]$ is a countably compact space of pointwise countable type that is not a paracompact $M$-space. Other examples regarding [6, Table 1] are mentioned in [6] and [8].

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REFERENCES


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