A NOTE ON THE CONCORDANCE HOMOTOPY GROUP OF REAL PROJECTIVE SPACE

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Abstract. By means of the mapping torus construction the following theorem is proved.

Theorem. If \( r \equiv 3 \mod 4 \) and \( r \geq 7 \), and \( \mathcal{P}_r \) is a homotopy \( P_r \), then there is an isomorphism \( \pi_0 \text{Diff}^+ : \mathcal{P}_r \cong \pi_0 \text{Diff}^+ : P_r \).

1. Introduction. For a manifold \( M \), let \( \text{Diff}^+ (M) \) be the group of diffeomorphisms of \( M \) isotopic to diffeomorphisms leaving some nonempty open set fixed. Let \( \pi_0 \text{Diff}^+ : M \) denote that group factored by those concordant to the identity. Similarly, let \( \text{Diff}^+ (M, A) \) denote the subgroup of \( \text{Diff}^+ (M) \) of diffeomorphisms fixing \( A \), and let \( \pi_0 \text{Diff}^+ : (M, A) \) denote \( \text{Diff}^+ (M, A) \) factored by the subgroup of those concordant \( \mod A \) to the identity.

Let \( P_r \) denote real projective space of dimension \( r \). In [4], an author establishes an isomorphism \( \pi_0 \text{Diff}^+ : P_r \cong \pi_{r+1+k} (P_{\infty}/k) \) for \( r \equiv 11 \mod 16 \) and \( k = a_2^L - r - 1 \) with \( a \) a positive integer and \( L \) a large positive integer. Suppose \( M \) is a smooth closed, \((l - 1)\)-connected and oriented manifold of dimension \( n \) with \( l = \lfloor n/2 \rfloor \). Suppose \( \xi : M \to M \) is a free smooth involution; then there is an equivariant embedding \( (S^l, -1) \subset (M, \xi) \) producing an embedding \( P_l \subset M/\xi \), and \( v(M/\xi) | P_l = k \eta \) where \( \eta \in KO(P_l) \) is the reduced canonical line bundle. The integer \( k \) is well defined \( \mod c(l) \), where \( c(l) \) is the order of \( KO(P_l) \), and its class \( \mod c(l) \) is called the type of \( \xi \). For \( k \) and \( l \) even, let

\[
I_{2l}(k) = \{(M, \xi) | M \sim S^l \times S^l, \text{type } \xi = k \}/\sim,
\]

where \( \sim \) means homotopy equivalent, and \( \sim \) means orientation-preserving equivariantly diffeomorphic. From [3] we recall that \( I_{2l}(k) \) has a canonical group structure, and that for \( l \equiv 6 \mod 8 \) and \( k = -2l/\mod c(l) \) we have an isomorphism \( \pi_{2l+k} (P_{\infty}/k) \cong I_{2l}(k) \). Thus we have for such \( l \) and \( k \) the isomorphism \( \pi_0 \text{Diff}^+ : B_{2l-1} \cong I_{2l}(k) \). We will say that a homotopy \( P_r \) is a smooth closed \( r \)-manifold \( \mathcal{P}_r \), homotopy equivalent to \( P_r \). It is the object of this note to find a generalization, for \( \mathcal{P}_r \), a homotopy \( P_r \) with \( r \equiv 3 \mod 4 \), of the isomorphism \( \pi_0 \text{Diff}^+ : B_{2l-1} \cong I_{2l}(k) \) above.

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In [1], the first author introduces abelian groups $I_n(k)$ generalizing the $I_{l2}(k)$ above. Suppose $M$ is a smooth closed $n$-manifold homotopy equivalent to $S^{[n/2]} \times S^{[(n+1)/2]}$ and suppose $\xi : M \to M$ is a smooth free involution. Let $l = [n/2]$. We will say $(M, \xi)$ is admissible if there exist disjoint copies, $P, P' \subset M/\xi$ of $P_1$ such that $P' \subset M/\xi - P$ is a homotopy equivalence. With $n$ even, all free involutions are admissible, but when $n$ is odd some are excluded. Then set

$$I_n(k) = \{(M, \xi) | M \sim S^{[n/2]} \times S^{[(n+1)/2]}, \text{type } S = k, \xi \text{ admissible} \}/\sim.$$ 

The equivalence relation $\sim$ is the same as before when $k$ is even—orientation preserving equivariant diffeomorphism—but when $k$ is odd, it is only equivariant diffeomorphism. From [1] we recall that $I_n(k)$ has a canonical abelian group structure, provided $n > 6$; also from [1] we recall that there is an exact sequence of abelian groups

$$\cdots \to \mathcal{L}_{n+1}(Z_2, (-1)^k) \xrightarrow{\partial} I_n(k) \xrightarrow{p} \Omega_n(\lambda(l, k)) \xrightarrow{\partial} \mathcal{L}_n(Z_2, (-1)^k),$$ 

where $\mathcal{L}_n(Z_2, (-1)^k)$ is a certain quotient of the Wall surgery group $L_n(Z_2, (-1)^k)$, and $\Omega_n(\lambda(l, k))$ is a certain Lashof cobordism group. It follows, for example, that $I_n(k)$ is finitely generated.

Now we can state the main theorem of this note.

**Theorem 2.** If $r = 3 \mod 4$ and $r > 7$, and $\mathcal{P}_r$ is a homotopy $\mathcal{P}_r$, then there is an isomorphism $\pi_0 \operatorname{Diff}^+ : \mathcal{P}_r \cong I_{r+1}(k)$ where $k = -r - 1 \mod c(l)$.

**Corollary.** If $\mathcal{P}_r$ is a homotopy $\mathcal{P}_r$, $r = 3 \mod 4$ and $r > 7$, then $\pi_0 \operatorname{Diff}^+ : \mathcal{P}_r \cong \pi_0 \operatorname{Diff}^+ : \mathcal{P}_r$.

The theorem is an immediate consequence of the following theorem. If $\mathcal{P}_r$ is a homotopy $\mathcal{P}_r$, there is an embedding, for $m = [r/2]$, unique up to isotopy $P_{m-1} \subset \mathcal{P}_r$ such that $\pi_1(\mathcal{P}_{m-1}) \to \pi_1(\mathcal{P}_r)$ is an epimorphism. Let $N_r$ be a tubular neighborhood of $P_{m-1}$ in $\mathcal{P}_r$. Let $f : \pi_0 \operatorname{Diff}^+ : (\mathcal{P}_r, N_r) \to \pi_0 \operatorname{Diff}^+ : \mathcal{P}_r$ be the forgetful homomorphism. Then we have the following:

**Theorem 1.** Let $r > 5$ and let $\mathcal{P}_r$ be a homotopy $\mathcal{P}_r$. Then there is a homomorphism $\tau : \pi_0 \operatorname{Diff}^+ : (\mathcal{P}_r, N_r) \to I_{r+1}(k)$, where $k = -r - 1 \mod c(l)$, such that:

1. kernel $(\tau) \subset \text{kernel } (f)$,
2. $\tau(\text{kernel } (f)) \subset \partial \mathcal{L}_{r+2}(Z_2, (-1)^k)$,
3. $\tau$ is an epimorphism.

We continue to use the notation implicit above: Given $r$, we set $l = \lfloor (r + 1)/2 \rfloor$, $m = \lfloor r/2 \rfloor$, $c(l) = \text{order } \overrightarrow{KO}(P)$, $k = \text{class of } -r - 1 \mod c(l)$, and $\eta_r$ = canonical line bundle over $\mathcal{P}_r$. If $(M, N)$ is a smooth manifold pair, $\nu(N; M)$ denotes the normal bundle of $N$ in $M$; $\tau(M)$ denotes the tangent bundle of $M$. If $\mathcal{P}_r$ is a homotopy $\mathcal{P}_r$ we have again the embedding $P_{m-1} \subset \mathcal{P}_r$ and its tubular neighborhood $N_r \subset \mathcal{P}_r$. Since $\mathcal{P}_r$ is necessarily tangentially
homotopy equivalent to $P_r$ and since $r - (m - 1) = l + 1 > (m - 1) + 1$, we have that $N_r$ is a smooth embedding of the cell bundle associated with $(l + 1)\eta_{m-1}$. There is an obvious homomorphism $\pi_0 \text{Diff}^+ : (\mathcal{C}_r, N_r) \rightarrow \pi_0 \text{Diff}^+ : \mathcal{C}_r$.

To see that $f$ is an epimorphism we introduce a homomorphism $d : \pi_0 \text{Diff}^+ : \mathcal{C}_r \rightarrow \mathbb{Z}_2$ defined as follows: If $x \in \pi_0 \text{Diff}^+ : \mathcal{C}_r$, we may choose a representative $\varphi : \mathcal{C}_r \rightarrow \mathcal{C}_r$ of $x$ such that $\varphi$ fixes $\mathcal{P}_1$ where $\mathcal{P}_1 \subset \mathcal{P}_{m-1}$. Then $d\varphi : \nu(P_1 : \mathcal{C}_r) \rightarrow \nu(P_1 : \mathcal{C}_r)$ represents a well-defined element $d(x) \in \widetilde{KO}^{-1}(P_1) = \mathbb{Z}_2$, and $x \rightarrow d(x)$ is a homomorphism.

**Proposition 1.** $d : \pi_0 \text{Diff}^+ : \mathcal{C}_r \rightarrow \mathbb{Z}_2$ is trivial.

**Proof.** We are indebted for the proof to R. Z. Goldstein. As in the definition of $d$, let $\varphi$ represent $x$, such that $\varphi$ fixes $\mathcal{P}_1$. Let $g \in H^1(S^1 : \mathbb{Z}_2)$ and $g' \in H^1(P_1 : \mathbb{Z}_2)$ be the nontrivial elements. Let $x(\varphi) \in H^1(S^1 \times P_1 : \mathbb{Z}_2)$ be 0 if $d(x) = 0$ and $pr^* g$ if $d(x) = 1$. Let $y = pr^* g'$. Let $S^1 \times \varphi \mathcal{C}_r$ be the mapping torus of $\varphi$. Then $S^1 \times \mathcal{P}_1 \subset S^1 \times \varphi \mathcal{C}_r$, and we have that the Stiefel-Whitney class

$$\omega(\nu(S^1 \times P_1 : S^1 \times \varphi \mathcal{C}_r)) = (1 + x(\varphi)(1 + y)^{-1}.$$ 

On the other hand, $\varphi$ is homotopic to the identity, so $S^1 \times \varphi \mathcal{C}_r$ has the homotopy type of $S^1 \times P_1$ and $\omega(\tau(S^1 \times \varphi \mathcal{C}_r)|S^1 \times P_1) = (1 + y)^{-1}$, since $y^2 = 0$. The proposition is proved.

If $x \in \pi_0 \text{Diff}^+ : \mathcal{C}_r$, then there is a representative $\varphi$ that fixes $P_{m-1} \subset \mathcal{C}_r$. We would like to find a representative that fixes $N_r$. The representative $\varphi$ at most twists $N_r$ by an element $d'(\varphi) \in \widetilde{KO}^{-1}(P_{m-1})$.

**Proposition 2.** $f : \pi_0 \text{Diff}^+ : (\mathcal{C}_r, N_r) \rightarrow \pi_0 \text{Diff}^+ : \mathcal{C}_r$ is an epimorphism.

**Proof.** The map $\widetilde{KO}^{-1}(P_{m-1}) \rightarrow \widetilde{KO}^{-1}(P_1)$ carries $d'(\varphi) \rightarrow d(x)$. This map is an isomorphism for $m \neq 0 \mod 4$, so we are done in that case by Proposition 1. If $m = 0 \mod 4$, then $\widetilde{KO}^{-1}(P_{m-1}) \equiv \widetilde{KO}^{-1}(P_1)$ is onto with infinite cyclic kernel. Then $d'(\varphi) \neq 0$ implies that $\nu(S^1 \times P_{m-1} : S^1 \times \varphi \mathcal{C}_r)$ has a nontrivial rational Pontrjagin class in dimension $m$, which is impossible, and the proposition is proved.

Now we construct the homomorphism $\tau : \pi_0 \text{Diff}^+ : (\mathcal{C}_r, N_r) \rightarrow I_{+1}(k)$ for $r \geq 5$. Briefly, it is the mapping torus construction followed by 'surgery' of $S^1 \times N_r \cup S^1_+ \times \mathcal{C}_r$. We construct a smooth manifold triad $(X; \partial_0 X, \partial_1 X)$ such that $\partial X = \partial_0 X \cup \partial_1 X$ and $\partial \partial_0 X = \partial \partial_0 X = \partial_0 X \cap \partial_1 X$ as follows: $X = D^2 \times \mathcal{C}_r$. With $S^1_+$ and $S^1$ the right and left hemispheres, respectively, we set $\Gamma = \text{closure} (\mathcal{C}_r - N_r)$, and $\partial_0 X = S^1_+ \times \Gamma$, and $\partial_1 X = S^1 \times N_r \cup S^1_+ \times P_1$. 

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Now, if \( x \in \pi_0 \text{Diff}^+ : (\mathcal{D}_r, \mathcal{N}_r) \) is represented by \( \varphi \), the mapping torus \( S^1 \times \mathcal{D}_r \) contains a codimension 0 submanifold canonically isomorphic to \( S^1 \times \mathcal{N}_r \cup S^1 \times \mathcal{D}_r \). Thus we may construct a well-defined surgery' with \((X; \partial_0 X, \partial_1 X)\) in place of the usual \( (D^{r+1}; S^r \times D^m, D^{r+1} \times S^{m-1})\): Set \( Y = \left((S^1 \times \mathcal{D}_r) \times [0,1]\right) \cup X \) where \( \partial_1 X \) is identified with \( (S^1 \times \mathcal{N}_r \cup S^1 \times \mathcal{D}_r) \times 1 \) by means of the canonical diffeomorphism. Then \( \partial Y = \partial_0 Y \amalg \partial_1 Y \) with \( \partial_0 Y = (S^1 \times \mathcal{D}_r) \times 0 \) and \( \partial_1 Y \) the other component of \( \partial Y \). It is routine to check that \( \partial_1 Y \) is the orbit manifold of a representative of an element \( \tau(\varphi) \in I_{r+1}(k) \). This element \( \tau(\varphi) \) is well defined. If \( \varphi' \) is another representative of \( x \), there is a concordance fixed on \( \mathcal{N}_r \) from \( \varphi \) to \( \varphi' \). Constructing \( Y' \) for \( \varphi' \) as above, and a similar manifold for the concordance, we obtain an \( h \)-cobordism finally from \( \partial_1 Y \) to \( \partial_1 Y' \) so that \( \tau(\varphi') = \tau(\varphi') \). Thus the map \( \tau: \pi_0 \text{Diff}^+: (\mathcal{D}_r, \mathcal{N}_r) \to I_{r+1}(k) \) is well defined by \( \tau(x) = \tau(\varphi) \) for \( \varphi \) a representative of \( x \).

To see that \( \tau \) is a homomorphism, we describe \( \tau \) a different way. Recall from [1] that the orbit space \( Q \) of an element of \( I_{r+1}(k) \) is obtained by gluing two copies of \( E(m\eta_{l+1}) \) (the cell bundle associated with \( m\eta_{l+1} \)) by means of a diffeomorphism \( \varphi': \partial E(m\eta_{l+1}) \to \partial E(m\eta_{l+1}) \). Since an \( (m+1) \)-plane bundle over \( F \) admitting a nonzero section and stably equivalent to \( m\eta_{l+1} \) is uniquely determined up to bundle equivalence, we have a diffeomorphism \( \Gamma \times 0 \cup \Gamma \times 1 \cong \partial E(m\eta_{l+1}) \) where \( \partial \Gamma \times 0 \) is glued to \( \partial \Gamma \times 1 \) by the identity. We have obvious homomorphisms

\[
\pi_0 \text{Diff}^+: (\mathcal{D}_r, \mathcal{N}_r) \to \pi_0 \text{Diff}^+: (\Gamma, \partial \Gamma) \to \pi_0 \text{Diff}^+: (\Gamma \times 0 \cup \Gamma \times 1, \Gamma \times 1) \to \pi_0 \text{Diff}^+: E(m\eta_{l+1}).
\]

If \( x \) is represented by \( \varphi \), and \( \varphi \to \varphi' \) under the above composition, it is straightforward to check that \( E(m\eta_{l+1}) \times 0 \cup \varphi E(m\eta_{l+1}) \times 1 \) represents \( \tau(x) \). Thus we have the commutative diagram:

\[
\begin{array}{ccc}
\pi_0 \text{Diff}^+: (\mathcal{D}_r, \mathcal{N}_r) & \xrightarrow{\tau} & \pi_0 \text{Diff}: \partial E(m\eta_{l+1}) \\
\downarrow & & \downarrow \\
I_{r+1}(k) & \xrightarrow{} &
\end{array}
\]

But the horizontal map is already a homomorphism, and according to [1] the vertical map is a homomorphism onto. It follows that \( \tau \) is a homomorphism.

**Proposition 3.** \( \text{kernel} \tau \subseteq \text{kernel} \phi \).

**Proof.** Suppose \( \tau(x) = 0 \) with \( \varphi \) a representative of \( x \). Then \( \tau(\varphi) \) has orbit space \( \partial E(m\eta_{l+2}) = \partial E \). Then we have \( S^1 \times \mathcal{D}_r = \partial (Y \cup E) \), where \( E \) is glued to \( Y \) along \( \partial Y_1 = \partial E \). We have \( S^1_+ \times \mathcal{D}_r \subset \partial (Y \cup E) \subset Y \cup E \), and
this composition of inclusions is a homotopy equivalence. Using an embedding $S^1_+ \times \mathbb{P}_r \times [0,1] \subset Y \cup E$ given by a boundary collar, an easy application of the relative $h$-cobordism theorem, as in [4], shows that there is a diffeomorphism $(S^1_+ \times \mathbb{P}_r, 1 \times \mathbb{P}_r) \cong (S^1 \times \mathbb{P}_r, 1 \times \mathbb{P}_r)$, which is the identity on the relative part. It follows that $f(x) = 0$, and Proposition 3 is proved.

**Proposition 4.** $\tau$ is an epimorphism.

**Proof.** Suppose $z \in I_{r+1}(k)$ has orbit space $Q$. We know $Q = E(m\eta_l + 1)$ or $E(m\eta_l + 1)$ for some diffeomorphism $\varphi: \partial E(m\eta_l + 1) \to \partial E(m\eta_l + 1)$; also $S^1_+ \times \Gamma = \partial_0 X$ is diffeomorphic to $E(m\eta_l + 1)$. Thus, the triad $(X; \partial_0 X, \partial_1 X)$ determines a surgery $Y$ from $\partial_1 Y = Q$ to $\partial_0 Y$. It is routine to check that $\partial_0 Y \cong S^1 \times \mathbb{P}_r$ (e.g. as in [4]). By Proposition 2, we may take $\varphi \in \pi_0 \text{Diff}: (\mathbb{P}_r, N_r)$, and clearly $\tau(\varphi) = y$. Proposition 4 is proved.

**Proof of Theorem 2.** We need only to check that

$$\tau(\ker f) \subset \partial \mathbb{E}_{r+2}(Z_2, (-1)^k).$$

Recall the definition of the Lashof cobordism group appearing in the exact sequence of [1]. First, $P[l, k] \to^\lambda(k) BO$ is a fibration such that $P_\infty \to P[l, k] \to^\lambda(k) BO$ is the $k$th Moore-Postnikov factorization of a map $P_\infty \to BO$ classifying $k \eta_\infty$. Then $\Omega_{r+1}(\lambda(l, k))$ is the $(r + 1)$st Lashof cobordism group defined by the fibration $\lambda(l, k)$. The map $p: I_{r+1}(k) \to \Omega_{r+1}(\lambda(l, k))$ is defined as follows: If $z \in I_{r+1}(k)$ has orbit space $Q$, and $P \subset Q$ is one of the canonical embeddings of $P_l$ in $Q$, then there is a commutative diagram

$$
\begin{array}{ccc}
P & \longrightarrow & P_\infty \\
\cap & & \searrow \\
Q & \longrightarrow & P[l, k] \\
\downarrow & & \downarrow \lambda(l, k) \\
& & BO
\end{array}
$$

with $Q \to BO$ a Gauss map. The obstructions are zero to finding a unique lift mod $P_\infty$ of $Q$ to $P[l, k]$. This lift represents an element of $\Omega_{r+1}(\lambda(l, k))$ which is well defined to be $p(z)$.

Let $x \in \ker f$ have representative $\varphi$ and let $\tau(x) = z \in I_{r+1}(k)$, and let $Y$ be the cobordism from $S^1 \times \mathbb{P}_r$ to $Q$, the orbit space of $z$. Since $x \in \ker f$, we have a diffeomorphism $S^1 \times \mathbb{P}_r \cong S^1 \times \mathbb{P}_r = \partial(D^2 \times \mathbb{P}_r)$. Gluing $D^2 \times \mathbb{P}_r$ to $Y$ via this diffeomorphism we obtain a manifold $\Lambda$, which may be written $\Lambda = (D^2 \times \mathbb{P}_r) \times 0 \cup a(D^{2} \times \mathbb{P}_r) \times 1$ with gluing map an embedding $\alpha: (S^1 \times N_r \cup S^1_+ \times \mathbb{P}_r) \times 1 \subset (S^1 \times \mathbb{P}_r) \times 0$ such that $\alpha((t, \xi), 1) = ((t, \xi), 0)$ for $t \in S^1_+$. We consider the lifting problem set by the following diagram:
It can be solved iff the lifting problem set by the following diagram can be solved:

$$\begin{align*}
(S^1 \times N_r \cup S^1_+ \times \mathbb{R}_r) \times 1 & \xrightarrow{\beta} P_\infty \\
\cap & \\
(D^2 \times \mathbb{R}_r) \times 1 & \rightarrow P[l, k]
\end{align*}$$

where $(S^1 \times N_r \cup S^1_+ \times \mathbb{R}_r) \times 1 \rightarrow P_\infty$ is $\alpha$ followed successively by projection $(S^1 \times \mathbb{R}_r) \times 0 \rightarrow \mathbb{R}_r$ and then $\mathbb{R}_r \rightarrow P_\infty$. But $\varphi \in \text{Diff}^+ \mathbb{R}_r$ implies $\varphi$ homotopic to the identity so there is a homotopy commutative diagram:

$$\begin{align*}
(S^1 \times N_r \cup S^1_+ \times \mathbb{R}_r) \times 1 & \xrightarrow{\beta} P_\infty \\
\cap & \\
(D^2 \times \mathbb{R}_r) \times 1 & \rightarrow P[l, k]
\end{align*}$$

From this diagram follows the solution of the second lifting problem, and so of the first. Let $\nu: \Lambda \rightarrow P[l, k]$ be that solution. Then $\nu|Q$ represents $p(z)$, and thus $0 = p(z) = p(\tau(x))$. From the exact sequence of [1], it follows that $\tau(x) \in \partial \mathcal{E}_{r+2}(Z_2, (−1)^k)$, the proof of Theorem 2 is complete.

Theorem 1 is an immediate consequence of Theorem 2 and the fact that $\mathcal{E}_s(z_2, +1) = 0$ for $s = 1 \mod 4$ [2].

**Bibliography**


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