REGULARITY OF SOLUTIONS TO AN ABSTRACT INHOMOGENEOUS LINEAR DIFFERENTIAL EQUATION

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Abstract. Let $T(t), t > 0,$ be a strongly continuous semigroup of linear operators on a Banach space $X$ with infinitesimal generator $A$ satisfying $T(t)X \subseteq D(A)$ for all $t > 0$. Let $f$ be a function from $[0, \infty)$ to $X$ of strong bounded variation. It is proved that $u(t) = \int_0^t T(t-s)f(s)\,ds$, $x \in X$, is strongly differentiable and satisfies $du(t)/dt = Au(t) + f(t)$ for all but a countable number of $t > 0$.

1. Introduction. Let $T(t), t > 0,$ be a strongly continuous semigroup of bounded linear operators on the Banach space $X$ with infinitesimal generator $A$ and let $f$ be an $X$-valued function on $[0, \infty)$. Our objective is to establish sufficient conditions so that the function

\begin{equation}
(1.1) \quad u(t) = T(t)x + \int_0^t T(t-s)f(s)\,ds, \quad x \in X,
\end{equation}

is a strong solution of the inhomogeneous linear differential equation

\begin{equation}
(1.2) \quad du(t)/dt = Au(t) + f(t), \quad u(0) = x.
\end{equation}

It is well known that $u(t)$ satisfies (1.2) for $t > 0$ provided that $x \in D(A)$ and $f$ is continuously differentiable (see [4, Theorem 1.19, p. 486] or [5, Theorem 6.5, p. 135]). It is also well known that $u(t)$ satisfies (1.2) for $t > 0$ provided that $x \in X$, $T(t), t > 0$, is homomorphic, and $f$ is Hölder continuous (see [4, Theorem 1.27, p. 491] or [5, Theorem 6.7, p. 138]). The theorem which we will prove demonstrates that $u(t)$ satisfies (1.2) under the assumptions that $T(t)X \subseteq D(A)$ for $t > 0$ and $f$ is of strong bounded variation. The main idea of our proof is to show that under our assumptions the integral in (1.1) lies in $D(A)$ and the image of this integral under $A$ may be represented as a Stieltjes integral.

Theorem. Suppose $T(t)X \subseteq D(A)$ for all $t > 0$ and $f$ is of strong bounded variation on $[0, r]$. For a given $x \in X$ let $u(t)$ be defined on $[0, r]$ by (1.1). Then, $u(t)$ satisfies the following:

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\begin{align*}
(1.3) & \quad u(t) \in D(A) \text{ for } t \in (0, r] \text{ and } Au(t) \text{ is continuous on } (0, r]; \\
(1.4) & \quad \frac{d^+ u(t)}{dt} = Au(t) + f(t^+) \text{ for all } t \in (0, r) \text{ and } \frac{d^+ u(t)}{dt} \text{ is continuous from the right on } (0, r); \\
(1.5) & \quad \frac{d^- u(t)}{dt} = Au(t) + f(t^-) \text{ for all } t \in (0, r] \text{ and } \frac{d^- u(t)}{dt} \text{ is continuous from the left on } (0, r]; \\
(1.6) & \quad \frac{d u(t)}{dt} = Au(t) + f(t) \text{ for all but a countable number of points in } [0, r) \text{ and } \frac{d u(t)}{dt} \text{ is continuous at all but a countable number of points in } [0, r].
\end{align*}

Before proving our theorem we first state some facts about Banach space-valued functions of strong bounded variation.

2. Vector-valued functions of strong bounded variation. Suppose \( f \) is of strong bounded variation from \([0, r]\) to \( X \) (according to the definition of \([3, p. 59]\)). The following properties of \( f \) may be proved analogously to the case of real-valued functions of bounded variation (for a discussion of real-valued functions of bounded variation the reader is referred to \([9, \text{ Chapter 2}] \) or \([2, \text{ Chapter II}] \):

\begin{align*}
(2.1) & \quad f \text{ has a right limit at each } t \in [0, r), \text{ denoted by } f(t^+), \text{ and } f(\cdot^+) \text{ is right continuous on } [0, r]; \\
(2.2) & \quad f \text{ has a left limit at each } t \in (0, r], \text{ denoted by } f(t^-), \text{ and } f(\cdot^-) \text{ is left continuous on } (0, r]; \\
(2.3) & \quad f(\cdot^-) \text{ is of strong bounded variation on } [0, r] \text{ (where for convenience we define } f(0^-) = f(0)), \text{ and if we define } v(t) \text{ to be the total variation of } f(\cdot^-) \text{ between } 0 \text{ and } t, \text{ then } v \text{ is nondecreasing and left continuous on } (0, r]; \\
(2.4) & \quad f \text{ is bounded on } [0, r] \text{ and continuous at all but a countable number of points in } [0, r].
\end{align*}

3. Proof of the theorem. We first prove the lemmas below, each of which is under the hypothesis of the theorem. In what follows we will suppose that \( M \)
is a constant such that $|T(t)| < M$ for $0 < t < r$ (see [4, p. 484]) and $\nu$ is defined as in (2.3).

**Lemma 3.1.** If $0 < t < r$, then

$$
\int_{0}^{t} T(t - s)f(s)ds \in D(A);
$$

(3.1)

$$
A\int_{0}^{t} T(t - s)f(s)ds = \int_{0}^{t}dT(t - s)f(s - ).
$$

**Proof.** Let $0 < t < r$. We observe that $\int_{0}^{t} T(t - s)f(s)ds$, the Riemann integral, exists since the integrand is bounded and continuous almost everywhere by virtue of (2.1) and the continuity properties of $T(t)$, $t > 0$. The function $T$ from $[0, t]$ to $B(X, X)$ (where $B(X, X)$ denotes the Banach space of bounded linear operators on $X$) is bounded on $[0, t]$. Further, since $T(t)X \subset D(A)$ for $t > 0$, $T$ is continuous from $[0, t]$ to $B(X, X)$ (see [3, Theorem 10.3.5, p. 310]). By (2.3) the set of discontinuities of $T(t - s)$, considered as a function of $s$ in $[0, t]$ to $B(X, X)$, has $\nu$ measure 0. That is, $s \rightarrow T(t - s)$ is discontinuous only at $t$ and, by (2.3), $\lim_{s\rightarrow t-} \nu(s) = \nu(t)$.

Thus, the Riemann-Stieltjes integral $\int_{0}^{t}dT(t - s)f(s - )$ exists in the sense that for each $\epsilon > 0$ there exists $\delta > 0$ such that if $\{s_i\}_{i=0}^{n}$ is a chain from 0 to $t$ such that sup$_{i=1, \ldots, n}|s_i - s_{i-1}| < \delta$, and $s_{i-1} < s_i < s_i$, then

$$
\left\| \sum_{i=1}^{n} (T(t - s_i) - T(t - s_{i-1}))f(s_i - ) - \int_{0}^{t}dT(t - s)f(s - ) \right\| < \epsilon
$$

(3.2)

(see [2, Theorem 13.16, p. 65 and Theorem 11.7, p. 53]).

For each positive integer $n$ let $s_i^n = it/n$, where $i = 0, 1, \ldots, n$. Define $g_n: [0, t] \rightarrow X$ by $g_n(s) = T(t - s)f(s_i^n - )$, where $s_{i-1} < s_i < s_i^n$, $i = 1, \ldots, n$, and $g_n(0) = T(t)f(0)$. By (2.4), $\{g_n\}$ is bounded on $[0, t]$ and $\{g_n\}$ converges to $T(t - s)f(s)$ almost everywhere on $[0, t]$. By the Lebesgue theorem,

$$
\lim_{n \rightarrow \infty} \int_{0}^{t} g_n(s)ds = \lim_{n \rightarrow \infty} \sum_{i=1}^{n} \int_{s_i^{n-1}}^{s_i^n} T(t - s)f(s_i^n - )ds = \int_{0}^{t} T(t - s)f(s)ds
$$

(3.3)

(see [3, Theorem 3.7.9, p. 83]). From [4, p. 486], $\int_{0}^{t}g_n(s)ds \in D(A)$ and

$$
A\int_{0}^{t} g_n(s)ds = \sum_{i=1}^{n} (T(t - s_i^n) - T(t - s_{i-1}^n))f(s_i^n - ).
$$

(3.4)

Then, by (3.2), (3.3), (3.4), and the closedness of $A$ we obtain (3.1).

**Lemma 3.2.** $A\int_{0}^{t} T(t - s)f(s)ds$ is continuous from the right in $t$ on $[0, r]$.

**Proof.** Let $0 < t < r$. First, we show that

$$
\lim_{h \rightarrow 0^+} A\int_{t}^{t+h} T(t + h - s)f(s)ds = 0.
$$

(3.5)

We observe that an argument similar to that of Lemma 3.1 shows that
\[ \int_t^{t+h} T(t + h - s)f(s)ds \in D(A) \]

and

\[ A\int_t^{t+h} T(t + h - s)f(s)ds = \int_t^{t+h} dT(t + h - s)f(s -). \]

Take \( h > 0 \) and sufficiently small. If \( \varepsilon > 0 \) there is a chain \( \{ s_i \}_{i=0}^n \) from \( t \) to \( t + h \) such that

\[ \left\| \int_t^{t+h} (T(t) - T(t + h - s)) df(s -) \right\| < \left\| \sum_{i=1}^n (T(t) - T(t + h - s_i))(f(s_i) - f(s_{i-1})) \right\| + \varepsilon \]

(3.6)

\[ < 2M \sum_{i=1}^n \| f(s_i) - f(s_{i-1}) \| + \varepsilon \]

\[ < 2M (\nu(t + h) - \nu(s_1)) + \varepsilon. \]

Then, (3.6) yields

\[ \left\| \int_t^{t+h} (T(t) - T(t + h - s)) df(s -) \right\| < 2M \left( \nu(t + h) - \lim_{s \to t^+} \nu(s) \right). \]

(3.7)

An integration by parts (see [2, Theorem 11.7, p. 53]) together with (3.7) yields

\[ \left\| A\int_t^{t+h} T(t + h - s)f(s)ds \right\| = \left\| \int_t^{t+h} dT(t + h - s)f(s -) \right\| \]

\[ = \left\| -\int_t^{t+h} T(t + h - s) df(s -) + f((t + h) -) - T(h)f(t -) \right\| \]

(3.8)

\[ = \left\| \int_t^{t+h} (T(t) - T(t + h - s)) df(s -) \right\| \]

\[ + f((t + h) -) - T(h)f((t + h) -) \]

\[ < 2M \left( \nu(t + h) - \lim_{s \to t^+} \nu(s) \right) + \|(I - T(h))f((t + h) -)\|. \]

In order to establish (3.5) we need only show that

(3.9)

\[ \lim_{h \to 0^+} \|(I - T(h))f((t + h) -)\| = 0. \]

But (3.9) holds by virtue of the fact that the range of \( f(\cdot -) \) on \([0, r]\) lies in a compact set of \( X \) and \( \lim_{h \to 0^+} (I - T(h))z = 0 \) uniformly for \( z \) in a compact set. The right continuity of \( A\int_0^T(t - s)f(s)ds \) in \( t \) now follows from (3.5) and the fact that
\[ A \int_0^{t+h} T(t + h - s) f(s) ds - A \int_0^t T(t - s) f(s) ds \]

\[ = (T(h) - I) A \int_0^t T(t - s) f(s) ds + A \int_t^{t+h} T(t + h - s) f(s) ds. \]

**Lemma 3.3.** \( A \int_0^t T(t - s) f(s) ds \) is continuous from the left in \( t \) on \( (0, r) \).

**Proof.** Let \( 0 < t < r \). Observe that for \( c > 0 \) and sufficiently small,

\[
\begin{align*}
A \int_{t-c}^t T(t - s) f(s) ds &= \int_{t-c}^t dT(t - s) f(s -) \\
&= \int_{t-c}^t T(t - s) df(s -) + f(t -) - T(c) f((t - c) -) \\
&< M(v(t) - v(t - c)) + M\|f(t -) - f((t - c) -)\| \\
&+ \|(I - T(c)) f(t -)\|.
\end{align*}
\]

If \( h > 0 \) and \( c > 0 \) are both sufficiently small, then (3.10) applied twice below yields

\[
\begin{align*}
A \int_0^t T(t - s) f(s) ds - A \int_0^{t-h} T(t - h - s) f(s) ds &= AT(c) \left( \int_0^{t-h-c} (T(h) - I) T(t - h - c - s) f(s) ds \\
&+ \int_{t-h-c}^{t-c} T(t - c - s) f(s) ds \right) \\
&+ A \int_{t-c}^t T(t - s) f(s) ds - A \int_{t-h-c}^{t-h} T(t - h - s) f(s) ds \\
< |AT(c)| \left( \int_0^{t-h-c} (T(h) - I) T(t - h - c - s) f(s) ds \right) \\
&+ Mh \sup_{s \in [t-h-c, t-c]} \|f(s)\| \\
&+ M(v(t) - v(t - c)) + M\|f(t -) - f((t - c) -)\| \\
&+ \|(I - T(c)) f(t -)\| + M(v(t - h) - v(t - h - c)) \\
&+ M\|f((t - h) -) - f((t - h - c) -)\| \\
&+ \|(I - T(c)) f((t - h) -)\|.
\end{align*}
\]

For a given \( \varepsilon > 0 \) first choose \( c > 0 \) and then choose \( \delta > 0 \) such that if \( 0 < h < \delta \), then (3.11) is \( < \varepsilon \) (use the fact that \( v \) and \( f(\cdot -) \) are left continuous at \( t \) and \( \lim_{h \to 0+} (T(h) - I) z = 0 \) uniformly for \( z \) in a compact set). The left continuity of \( A \int_0^t T(t - s) f(s) ds \) then follows immediately.

To complete the proof of the theorem we see that (1.3) follows from
Lemmas 3.1, 3.2, and 3.3. To prove (1.4) let $0 < t < r$ and observe that for
$h > 0$ and sufficiently small we have
\[
(u(t + h) - u(t))/h = (T(t + h)x - T(t)x)/h
+ \frac{1}{h} \int_t^{t+h} T(t + h - s)f(s)ds
+ \frac{T(h) - I}{h} \int_0^t T(t - s)f(s)ds.
\]
By (2.1)
\[
\lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} T(t + h - s)f(s)ds = f(t + )
\]
and (1.4) then follows from Lemmas 3.1 and 3.2. To prove (1.5) let $0 < t < r$
and observe that for $h > 0$ and sufficiently small we have
\[
(u(t - h) - u(t))/(-h) = (T(t - h)x - T(t)x)/(-h)
+ \frac{1}{h} \int_{t-h}^t T(t - s)f(s)ds
+ \frac{T(h) - I}{h} \int_{-h}^0 T(t - h - s)f(s)ds.
\]
By (2.2)
\[
\lim_{h \to 0^+} \frac{1}{h} \int_{t-h}^t T(t - h - s)f(s)ds = f(t - ).
\]
Denote
\[
z(h) \overset{\text{def}}{=} \int_0^{-h} T(t - h - s)f(s)ds.
\]
By Lemma 3.1, $z(h) \in D(A)$ and by Lemma 3.3,
\[
\lim_{h \to 0^+} \frac{T(h) - I}{h} z(h) = \lim_{h \to 0^+} \frac{1}{h} \int_0^h T(s)Az(h)ds = Az(0),
\]
which yields (1.5). Finally, (1.6) follows immediately from (1.4), (1.5), and
(2.4).
We conclude with the observation that our theorem may be applied to
nonlinear evolution equations of the form $du(t)/dt = Au(t) + B(u(t))$. If
$-B$ is an accretive continuous everywhere defined nonlinear operator on $X$,
then there exists a solution $u(t)$ to the Volterra integral equation
\[
u(t) = T(t)x + \int_0^t T(t - s)B(u(s))ds, \quad x \in X
\]
(see [10, Theorem I]). If we assume that $x \in D(A)$, then it can be shown that
$u(t)$ is Lipschitz continuous. If we also assume that $B$ is Lipschitz continuous
and $T(t)X \subset D(A)$ for all $t > 0$, then our theorem implies $u(t)$ satisfies
$du(t)/dt = Au(t) + B(u(t))$ for all $t > 0$. If it is not true that $T(t)X \subset
D(A)$, then this conclusion may not hold (see [10, Example 4.1]). A similar
observation is made in [7] for the case that $T(t), t \geq 0$, is a holomorphic semigroup.

References


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