MULTIPLIERS ON DUAL $A^*$-ALGEBRAS

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Abstract. Let $A$ be an $A^*$-algebra which is a dense $^*$-ideal of a $B^*$-algebra $\mathfrak{A}$. We use tensor products and the algebra $M_r(A)$ of left multipliers on $A$ to obtain a characterization of duality in $A$. We show, moreover, that if $A$ is dual then $M_r(A)$ is algebra isomorphic to the second conjugate space $\mathfrak{A}^{**}$ of $\mathfrak{A}$ when $\mathfrak{A}^{**}$ is given Arens product.

1. Introduction. Let $A$ be an $A^*$-algebra which is a dense $^*$-ideal of a $B^*$-algebra $A$. In [8] a necessary and sufficient condition was given for $A$ to be dual which was expressed in terms of the weak operator topology on $M_r(A)$, the algebra of right multipliers on $A$, and a certain property of $A$ called property (P2). In this paper we give several characterizations of property (P2) and then use some of them to give conditions for duality in $A$. Our presentation here differs somewhat from that in [8]. We use the tensor product approach as developed in [3] and [6]. Particularly in §2 we follow closely the presentation given in [3].

We shall use the notation of [8]. An $A^*$-algebra $A$ is said to be of the first kind if it is an ideal of its completion $\mathfrak{A}$ in the auxiliary norm $|\cdot|$. It follows that there exists a constant $k > 0$ such that $\|xy\| < k\|x\|\|y\|$ for all $x \in A, y \in \mathfrak{A}$ [4, Lemma 4, p. 18]. If $A$ is a modular annihilator $A^*$-algebra then $|\cdot|$ is unique [1, (1.3), p. 6] so that $\mathfrak{A}$ is also unique.

2. The property (P2). Let $A$ be a Banach algebra, $A^*$ and $A^{**}$ its first and second conjugate spaces. Let $A \hat{\otimes} A^*$ be the projective tensor product of $A$ and $A^*$ [7, pp. 92–95]. Then $A \hat{\otimes} A^*$ is a Banach space with elements of the form $\sum_{k=1}^{\infty}a_k \hat{\otimes} f_k$ such that $\sum_{k=1}^{\infty}\|a_k\|\|f_k\| < \infty$, $a_k \in A, f_k \in A^*$, and the norm given by

$$\|h\| = \inf \left\{ \sum_{k=1}^{\infty}\|a_k\|\|f_k\| : h = \sum_{k=1}^{\infty}a_k \hat{\otimes} f_k \right\}.$$

For $a \in A, f \in A^*$, let $af \in A^*$ be given by $(af)x = f(xa)$, $x \in A$. This makes $A^*$ into a left Banach $A$-module. We note that if $\sum_{k=1}^{\infty}a_k \hat{\otimes} f_k \in A \hat{\otimes} A^*$, then $\sum_{k=1}^{\infty}a_kf_k \in A^*$. Let $\psi$ be the continuous linear map of $A \hat{\otimes} A^*$ into $A^*$ given by
\[ \psi(a \otimes f) = af \quad (a \in A, f \in A^*). \]

For \( a \in A, F \in A^{**} \), let \( Fa \in A^{**} \) be given by \((Fa)f = F(af), f \in A^*\). Then \( A^{**} \) is a right Banach \( A \)-module. (See [6] for the definition and properties of Banach \( A \)-modules.)

Let \( A \odot A^* \) be the Banach space \( A \otimes A^*/\ker(\psi) \) with the usual quotient norm, where \( \ker(\psi) \) is the kernel of \( \psi \). Then \( (A \odot A^*)^* \) consists of all those \( \mathcal{F} \in (A \otimes A^*)^* \) which vanish on \( \ker(\psi) \). Now, for each \( F \in A^{**} \), let \( \mathcal{F}_F \in (A \otimes A^*)^* \) be given by
\[
\mathcal{F}_F(a \otimes f) = F(\psi(a \otimes f)) = F(af) \quad (a \in A, f \in A^*).
\]

Each \( \mathcal{F}_F \) vanishes on \( \ker(\psi) \), so that \( \{ \mathcal{F}_F : F \in A^{**} \} \) may be identified as a subspace of \( (A \odot A^*)^* \). Moreover if \( Fa = 0 \) for all \( a \in A \) implies \( F = 0 \), then \( F \to \mathcal{F}_F \) is a one-to-one map of \( A^{**} \) into \( (A \otimes A^*)^* \). Thus in this case \( F \to \mathcal{F}_F \) identifies \( A^{**} \) as a subspace of \( (A \odot A^*)^* \).

Let \( \sigma \) denote the \( w^* \)-topology of \( (A \otimes A^*)^* \).

**Lemma 2.1.** \((A \odot A^*)^* \) is the \( \sigma \)-closure of \( \{ \mathcal{F}_F : F \in A^{**} \} \).

**Proof.** We have
\[
\ker(\psi) = \left\{ \sum_{k=1}^{\infty} a_k \otimes f_k \in A \otimes A^* : \sum_{k=1}^{\infty} a_k f_k = 0 \right\} = \left\{ \sum_{k=1}^{\infty} a_k \otimes f_k \in A \otimes A^* : F\left(\sum_{k=1}^{\infty} a_k f_k\right) = 0, F \in A^{**} \right\}.
\]

Thus \( \ker(\psi) = \bigcap \{ \ker(\mathcal{F}_F) : F \in A^{**} \} \), which means that \( \ker(\psi) \) is the polar of \( \{ \mathcal{F}_F : F \in A^{**} \} \). Therefore, by the Bipolar Theorem [7, p. 126], \( (A \odot A^*)^* \) is the \( \sigma \)-closure of \( \{ \mathcal{F}_F : F \in A^{**} \} \). This completes the proof.

We observe that if \( A^2 = (0) \), then \( \ker(\mathcal{F}_F) = A \otimes A^* \), for every \( F \in A^{**} \), so that \( \ker(\psi) = A \otimes A^* \) and consequently \( (A \odot A^*)^* = (0) \).

Let \( \mathcal{B}(A, A^{**}) \) be the Banach space of all bounded linear operators \( T : A \to A^{**} \) normed with the operator bound norm. For each \( \mathcal{F} \in (A \otimes A^*)^* \), let \( T_\mathcal{F} \) be the map on \( A \) into \( A^{**} \) given by
\[
(f, T_\mathcal{F}(a)) = \mathcal{F}(a \otimes f) \quad (a \in A, f \in A^*).
\]

Then clearly \( T_\mathcal{F} \in \mathcal{B}(A, A^{**}) \) for every \( \mathcal{F} \in (A \otimes A^*)^* \), and it is easy to check that the map \( \phi : \mathcal{F} \to T_\mathcal{F} \) is an isometric isomorphism of \( (A \otimes A^*)^* \) onto \( \mathcal{B}(A, A^{**}) \). Give \( \mathcal{B}(A, A^{**}) \) the image of the \( \sigma \) topology by the map \( \phi \). Following Maté [3], we shall refer to this topology as the ultraweak topology on \( \mathcal{B}(A, A^{**}) \).

Now consider \( A^{**} \) as a right Banach \( A \)-module and let \( \text{Hom}_A(A, A^{**}) \) be the set of all \( T \in \mathcal{B}(A, A^{**}) \) such that \( T(ab) = T(a)b, a, b \in A \). The canonical map \( \pi : A \to A^{**} \) belongs to \( \mathcal{B}(A, A^{**}) \) since \( \pi(ab)f = f(ab) = \pi(a)(bf) \) for all \( a, b \in A \) and \( f \in A^* \). For each \( F \in A^{**} \), let \( T_F : A \to A^{**} \) be given by \( T_F(a) = Fa, a \in A \). Then \( T_F \in \text{Hom}_A(A, A^{**}) \), and we have \( \phi(\mathcal{F}_F) \)
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$T_F$ for all $F \in A^{**}$. In view of Lemma 2.1 and the fact that the ultraweak closure of \{ $T_F; F \in A^{**}$ \} $\subseteq$ Hom$_A(A, A^{**})$ we have

**Lemma 2.2.** $\phi(A \circ A^*) \subseteq$ Hom$_A(A, A^{**})$ and is the ultraweak closure of \{ $T_F; F \in A^{**}$ \}.

If $\phi$ maps $(A \circ A^*)^*$ onto Hom$_A(A, A^{**})$, we shall write $(A \circ A^*)^* \cong$ Hom$_A(A, A^{**})$. In this case, for every $T \in$ Hom$_A(A, A^{**})$, $\widetilde{T}_F \in (A \hat{\otimes} A^*)^*$, given by $\widetilde{T}_F(a \hat{\otimes} f) = (f, T(a))$, belongs to $(A \circ A^*)^*$. In particular,

$\widetilde{T}_a(a \hat{\otimes} f) = f(a)$, for all $a, f \in A^*$.

We recall that a Banach algebra $A$ is said to have property (P2) if:

$a_k \in A$, $f_k \in A^*$, $\sum_{k=1}^{\infty} \| a_k \| \| f_k \| < \infty$ and $\sum_{k=1}^{\infty} a_k f_k = 0$ implies that $\sum_{k=1}^{\infty} f_k(a_k) = 0$. (This is the left-hand version of the definition given in [8].)

**Theorem 2.3.** Let $A$ be a Banach algebra. Then the following statements are equivalent:

(i) $A$ has property (P2).

(ii) For $h = \sum_k a_k \hat{\otimes} f_k \in$ ker($\psi$) we have $\sum_{k=1}^{\infty} f_k(a_k) = 0$.

(iii) $\widetilde{T}_a$ vanishes on ker($\psi$).

(iv) $\widetilde{T}_a \in (A \circ A^*)^*$.

(v) Hom$_A(A, A^{**}) \cong (A \circ A^*)^*$.

(vi) There exists a net \{ $u_a$ \} in $A$ such that \{ $\widetilde{T}_{\sigma(u_a)}$ \} converges to $\widetilde{T}_a$ in the w*-topology on $(A \hat{\otimes} A^*)^*$.

**Proof**. (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv) are clear.

(iv) $\Rightarrow$ (vi). Suppose (iv) holds. Then ker($\widetilde{T}_a$) $\supset$ ker($\psi$). We have

$\widetilde{T}_a(h) = \sum_{k=1}^{\infty} a_k \hat{\otimes} f_k = \sum_{k=1}^{\infty} \widetilde{T}_a(a_k \hat{\otimes} f_k) = \sum_{k=1}^{\infty} f_k(a_k),$

for all $h = \sum_{k=1}^{\infty} a_k \hat{\otimes} f_k \in A \hat{\otimes} A^*$. Since \{ $\widetilde{T}_F; F \in A^{**}$ \} is $\sigma$-dense in $(A \circ A^*)^*$, there exists a net \{ $F_\alpha$ \} in $A^{**}$ such that $\widetilde{T}_{F_\alpha}(h) \rightarrow \widetilde{T}_a(h)$ for all $h \in A \hat{\otimes} A^*$. Since $\sigma(A)$ is w*-dense in $A^{**}$ and $\sigma$ is weaker than the w*-topology on $A^{**}$, it follows that \{ $\widetilde{T}_{\sigma(a)}; a \in A$ \} is $\sigma$-dense in $(A \circ A^*)^*$.

Hence there exists a net \{ $u_a$ \} in $A$ such that \{ $\widetilde{T}_{\sigma(u_a)}$ \} $\sigma$-converges to $\widetilde{T}_a$. We have

$\widetilde{T}_{\sigma(u_a)}(h) = \sigma(u_a)(\psi(h)) = \sum_{k=1}^{\infty} \sigma(u_a)(a_k f_k) = \left( \sum_{k=1}^{\infty} a_k f_k \right)(u_a) = \sum_{k=1}^{\infty} f_k(u_a a_k).$

Thus
(1) \[ \lim_{\alpha} \mathcal{F}_{\pi(u_\alpha)}(h) = \lim_{\alpha} \sum_{k=1}^{\infty} f_k(u_\alpha a_k) = \sum_{k=1}^{\infty} f_k(a_k) = \mathcal{F}_{\pi}(h), \]

for all \( h = \sum_{k=1}^{\infty} a_k \otimes f_k \in A \otimes A^*. \)

(vi) \( \Rightarrow \) (v). We have \( \phi((A \circ A^*)^*) \subseteq \text{Hom}_A(A, A^{**}) \). We need only show that \( \text{Hom}_A(A, A^{**}) \subseteq \phi((A \circ A^*)^*) \). Let \( T \in \text{Hom}_A(A, A^{**}) \) and let \( \mathcal{F}_T \) be the corresponding element of \( (A \otimes A^*)^* \). Then, using (1), we obtain (identifying \( A \) as a subset of \( A^{**} \) and \( A^* \) as a subset of \( A^{**} \)):

\[
\mathcal{F}_T \left( \sum_{k=1}^{\infty} a_k \otimes f_k \right) = \sum_{k=1}^{\infty} \mathcal{F}_T(a_k \otimes f_k) = \sum_{k=1}^{\infty} (f_k(T(a_k)))
\]

\[
= \sum_{k=1}^{\infty} (T^*(a_k))(a_k) = \lim_{\alpha} \sum_{k=1}^{\infty} (T^*(f_k))(u_\alpha a_k) = \lim_{\alpha} \sum_{k=1}^{\infty} f_k(T(a_\alpha) a_k)
\]

\[
= \lim_{\alpha} \sum_{k=1}^{\infty} (T^*(a_k f_k))(u_\alpha) = \sum_{k=1}^{\infty} T^*(a_k f_k)
\]

where \( T^* \) is the conjugate of \( T \). Hence if \( \sum_{k=1}^{\infty} a_k f_k = 0 \) then \( \mathcal{F}_T(\sum_{k=1}^{\infty} a_k \otimes f_k) = 0 \), so that \( \ker(T) \subseteq \ker(\mathcal{F}_T) \). Thus \( \text{Hom}_A(A, A^{**}) \subseteq \phi((A \circ A^*)^*) \) and so \( \text{Hom}_A(A, A^{**}) \cong (A \circ A^*)^* \).

(v) \( \Rightarrow \) (iv). This is clear since \( \pi \in \text{Hom}_A(A, A^{**}) \).

3. Dual \( A^* \)-algebras. Let \( A \) be a Banach algebra. A map \( T: A \to A \) is called a left (resp. right) multiplier if \( T(ab) = F(a)Z> \) (resp. \( T(ab) = aT(b) \)), for all \( a, b \in A \). Let \( M_f(A) \) (resp. \( M_r(A) \)) be the set of all bounded linear left (resp. right) multipliers on \( A \). \( M_f(A) \) and \( M_r(A) \) are Banach algebras under the usual operations for operators and the operator norm. We observe that if \( T \in M_f(A) \) then the composite map \( \tau \circ T \in \text{Hom}_A(A, A^{**}) \). Let \( \phi_{\pi} \) be the map of \( M_f(A) \) into \( \text{Hom}_A(A, A^{**}) \) given by

\[ \phi_{\pi}(T) = \tau \circ T \quad (T \in M_f(A)). \]

For any Banach space \( X \), let \( S(X) \) denote the closed unit ball of \( X \). It follows from the proof of [8, Theorem 4.7, p. 286] that if \( A \) is a dual \( A^* \)-algebra of the first kind then \( S(M_f(A)) \) is \( \tau_r \)-compact, where \( \tau_r \) is the weak operator topology on \( M_f(A) \). (We take the left-hand version of the arguments in [8, p. 286].)

**Theorem 3.1.** Let \( A \) be an \( A^* \)-algebra of the first kind. Then the following statements are equivalent:

(i) \( A \) is dual.

(ii) \( \phi_{\pi}(M_f(A)) \) is the ultraweak closure of \( \{T_F: F \in A^{**}\} \).

**Proof.** (i) \( \Rightarrow \) (ii). Suppose \( A \) is dual. Then, by [8, Theorem 4.7, p. 286], it has property (P2) and therefore, by Theorem 2.3, \((A \circ A^*)^* \cong \text{Hom}_A(A, A^{**})\). Hence \( \text{Hom}_A(A, A^{**}) \) is the ultraweak closure of \( \{T_F: F \in A^{**}\} \).
A**). Now \( T_F(a)f = (Fa)f = (F \cdot \pi(a))f \) and, by [9, Theorem 5.2, p. 830], \( \pi(A) \) is an ideal of \( A** \) when \( A** \) is given either Arens product, so that \( T_F(a) \in \pi(A) \) for all \( a \in A \). Therefore \( T_F = \pi \circ T \), for some \( T \in M_i(A) \), and so \( \{ T_F: F \in A** \} \subseteq \phi\pi(M_i(A)) \). Let \( Q \in \text{Hom}_A(A, A** \) Then, by Lemma 2.2 and the fact that \( (A \circ A^*)^* = \text{Hom}_A(A, A** \), there exists a net \( \{ F_\alpha \} \) in \( A** \) such that \( (f, T_{F_\alpha}(a)) \to (f, Q(a)) \) for all \( a \in A, f \in A^* \). Let \( T_\alpha \in M_i(A) \) be such that \( T_{F_\alpha} = \pi \circ T_\alpha \), for all \( \alpha \). Then \( (f, T_{F_\alpha}(a)) = f(T_\alpha(a)) \) since \( T_\alpha(a) \in A \). But, by [8, Theorem 4.7, p. 286], \( M_i(A) \) is \( \tau_r \)-complete. Hence there exists \( T \in M_i(A) \) such that \( f(T_\alpha(a)) \to (f(T(a)), \) for all \( a \in A, f \in A^* \). This shows that \( \pi(T(a))f = (f, Q(a)) \), for all \( a \in A, f \in A^* \), or equivalently, \( \pi(T(a)) = Q(a) \), for all \( a \in A \), i.e., \( Q = \pi \circ T \). Thus \( Q \in \phi\pi(M_i(A)) \) and so \( \text{Hom}_A(A, A** \) \( = \phi\pi(M_i(A)) \). Since \( (A \circ A^*)^* = \text{Hom}_A(A, A** \) and since \( (A \circ A^*)^* \) is the \( \sigma \)-closure of \( \{ \xi_F: F \in A** \) \), it follows that \( \phi\pi(M_i(A)) \) is the ultraweak closure of \( \{ T_F: F \in A** \) \).

(ii) \( \Rightarrow \) (i). Suppose (ii) holds. Then, in view of Lemma 2.2, \( M_i(A) \) is isometrically isomorphic to \( (A \circ A^*)^* \). From Lemma 2.1 we obtain \( \Sigma((A \circ A^*)^*) \) is \( \sigma \)-compact, and since \( \tau_r \) is weaker than the ultraweak topology on \( M_i(A) \), it follows that \( \Sigma(M_i(A)) \) is \( \tau_r \)-compact and therefore \( \tau_r \)-complete. Let \( I \) be the identity element of \( M_i(A) \). Since \( \{ T_F: F \in A** \) \) is ultraweak dense in \( \phi\pi(M_i(A)) \), there exists a net \( \{ F_\alpha \} \) in \( A** \) such that \( T_{F_\alpha} \) converges ultraweakly to \( \pi \circ I = \pi \), or equivalently, \( \xi_{F_\alpha} \sigma \)-converges to \( \xi_{\pi} \). Since \( \pi(A) \) is \( w^* \)-dense in \( A** \) and the \( w^* \)-topology is stronger than the \( \sigma \)-topology on \( A** \), it follows that there exists a net \( \{ u_\alpha \} \) in \( A \) such that \( \{ \xi_{\pi(u_\alpha)} \} \sigma \)-converges to \( \xi_{\pi} \) and so, by Theorem 2.3, \( A \) has property (P2). Therefore, by [8, Theorem 4.7, p. 287], \( A \) is dual.

**Corollary 3.2.** Let \( A \) be an \( A^* \)-algebra of the first kind. Then \( A \) is dual if and only if \( \phi\pi(M_i(A)) = \phi((A \circ A^*)^*) \).

**Corollary 3.3.** Let \( A \) be a modular annihilator \( A^* \)-algebra of the first kind. If \( \text{Hom}_A(A, A** \) is the ultraweak closure of \( \{ T_F: F \in A** \) \) then \( A \) is dual.

**Proof.** By [9, Theorem 5.2, p. 830], \( \pi(A) \) is an ideal of \( A** \) so that \( T_F \) maps \( A \) into \( \pi(A) \) for every \( F \in A** \). Hence if \( \text{Hom}_A(A, A** \) is the ultraweak closure of \( \{ T_F: F \in A** \) \), then \( \text{Hom}_A(A, A** \) \( = \phi\pi(M_i(A)) \) by the proof above. Therefore \( A \) is dual by Theorem 3.1.

4. A realization of the algebra \( M_i(A) \).

**Theorem 4.1.** Let \( A \) be a dual \( A^* \)-algebra of the first kind and let \( \mathfrak{A} \) be its completion. Let \( \pi_{\mathfrak{A}} \) be the canonical map of \( \mathfrak{A} \) into \( A** \). Then \( \pi_{\mathfrak{A}}(A) \) is an ideal of \( A** \) when \( A** \) is given Arens product.

**Proof.** Let \( x \in A, F \in \mathfrak{A}** \) and let \( \{ e_\alpha \} \) be a maximal orthogonal family of selfadjoint minimal idempotents in \( A \). By [4, Theorem 16, p. 30], \( \Sigma e_\alpha x \) is summable to \( x \) in the norm \( || \cdot || \), and hence there exists only a countable number of \( e_\alpha \) for which \( e_\alpha x \neq 0 \), say \( e_{\alpha_1}, e_{\alpha_2}, \ldots \). Since \( A \) and \( \mathfrak{A} \) have the
same socle and \( \pi_\mathcal{A}(\mathcal{A}) \) is an ideal of \( \mathcal{A}^{**} \), it follows that \( F \ast \pi_\mathcal{A}(e_{\alpha_i}) \in \pi_\mathcal{A}(A) \) for \( i = 1, 2, \ldots \). Let \( m, n \) be positive integers, \( m < n \). By [4, Lemma 4, p. 18], we have

\[
\left\| \sum_{i=1}^{n} F \ast \pi_\mathcal{A}(e_{\alpha_i})\pi_\mathcal{A}(x) - \sum_{i=1}^{m} F \ast \pi_\mathcal{A}(e_{\alpha_i}) \ast \pi_\mathcal{A}(x) \right\|
\leq k \left( \left\| \sum_{i=m+1}^{n} F \ast \pi_\mathcal{A}(e_{\alpha_i}) \right\| \left\| \sum_{i=m+1}^{n} \pi_\mathcal{A}(e_{\alpha_i}) \ast \pi_\mathcal{A}(x) \right\| \right)
\leq k \left\| F \right\| \left\| \sum_{i=m+1}^{n} \pi_\mathcal{A}(e_{\alpha_i}) \ast \pi_\mathcal{A}(x) \right\|
\leq k \left\| F \right\| \left\| \pi_\mathcal{A}(x) \right\| ,
\]

where \( |F| \) denotes the norm of \( F \) in \( \mathcal{A}^{**} \) and \( k \) is a positive constant. Thus \( \{ \sum_{i=1}^{n} F \ast \pi_\mathcal{A}(e_{\alpha_i}) \ast \pi_\mathcal{A}(x) \} \) is a Cauchy sequence in \( \pi_\mathcal{A}(A) \) with respect to the norm \( \| \cdot \| \), and so there exists \( z \in A \) such that \( \pi_\mathcal{A}(z) = \sum_{i=1}^{\infty} F \ast \pi_\mathcal{A}(e_{\alpha_i}) \ast \pi_\mathcal{A}(x) \). Since \( \sum_{i=1}^{\infty} F \ast \pi_\mathcal{A}(e_{\alpha_i}) \ast \pi_\mathcal{A}(x) \) also converges to \( \pi_\mathcal{A}(z) \) and to \( F \ast \pi_\mathcal{A}(x) \) in the norm \( \| \cdot \| \), we have \( \pi_\mathcal{A}(z) = F \ast \pi_\mathcal{A}(x) \). Hence \( F \ast \pi_\mathcal{A}(x) \in \pi_\mathcal{A}(A) \), for all \( x \in A \) and \( F \in \mathcal{A}^{**} \). Similarly we can show that \( \pi_\mathcal{A}(x) \ast F \in \mathcal{A}^{**} \), for all \( x \in A \) and \( F \in \mathcal{A}^{**} \). Therefore \( \pi_\mathcal{A}(A) \) is an ideal of \( \mathcal{A}^{**} \).

**Theorem 4.2.** Let \( A \) be a dual \( A^* \)-algebra of the first kind and \( \mathcal{A} \) its completion. Then \( M_1(A) \) is algebra isomorphic to \( \mathcal{A}^{**} \) when \( \mathcal{A}^{**} \) is given Arons product. This isomorphism is given by the following relation: For each \( T \in M_1(A) \) there exists a unique \( F_T \in \mathcal{A}^{**} \) such that \( \pi_\mathcal{A}(Tx) = F_T \ast \pi_\mathcal{A}(x) \) \( (x \in A) \).

**Proof.** For each \( x \in A \), let \( \| x \|_A' = \sup\{ \| xy \| : \| y \| \leq 1, y \in A \} \). Then \( \| \cdot \|_A' \) is a norm on \( A \) which is equivalent to \( \| \cdot \| \) [4, Theorem 18, p. 31]. Hence if \( T \in M_1(A) \) and \( x \in A \), then

\[
\| Tx \|_A' = \sup\{ \| T(x)y \| : \| y \| \leq 1, y \in A \}
= \sup\{ \| T(xy) \| : \| y \| \leq 1, y \in A \}
\leq \| T \| \sup\{ \| xy \| : \| y \| \leq 1, y \in A \}
\leq k' \| T \| \| x \|,
\]

where \( k' \) is a constant \( > 0 \). Thus \( |Tx| \leq k''|x| \) for all \( x \in A \) and some constant \( k'' > 0 \). Since \( A \) is dense in \( \mathcal{A} \), it follows that \( T \) has a unique bounded extension \( T' \) to \( \mathcal{A} \). Clearly \( T' \in M_1(\mathcal{A}) \). By [2, Corollary 3.2, p. 509], there exists a unique \( F_T \in \mathcal{A}^{**} \) such that \( \pi_\mathcal{A}(Tx) = F_T \ast \pi_\mathcal{A}(x) \) for all \( x \in A \).
Since, by Theorem 4.1, $\pi_r(A)$ is an ideal of $\mathfrak{A}^{**}$, we have that $T \rightarrow F_T$ is an algebra isomorphism of $M_1(A)$ onto $\mathfrak{A}^{**}$.

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