MULTIPLIERS ON DUAL $A^*$-ALGEBRAS

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Abstract. Let $A$ be an $A^*$-algebra which is a dense $*$-ideal of a $B^*$-algebra $\mathcal{A}$. We use tensor products and the algebra $M_r(A)$ of left multipliers on $A$ to obtain a characterization of duality in $A$. We show, moreover, that if $A$ is dual then $M_r(A)$ is algebra isomorphic to the second conjugate space $\mathcal{A}^{**}$ of $\mathcal{A}$ when $\mathcal{A}^{**}$ is given Arens product.

1. Introduction. Let $A$ be an $A^*$-algebra which is a dense $*$-ideal of a $B^*$-algebra $A$. In [8] a necessary and sufficient condition was given for $A$ to be dual which was expressed in terms of the weak operator topology on $M_r(A)$, the algebra of right multipliers on $A$, and a certain property of $A$ called property (P2). In this paper we give several characterizations of property (P2) and then use some of them to give conditions for duality in $A$. Our presentation here differs somewhat from that in [8]. We use the tensor product approach as developed in [3] and [6]. Particularly in §2 we follow closely the presentation given in [3].

We shall use the notation of [8]. An $A^*$-algebra $A$ is said to be of the first kind if it is an ideal of its completion $\mathcal{A}$ in the auxiliary norm $| \cdot |$. It follows that there exists a constant $k > 0$ such that $\|xy\| \leq k\|x\|\|y\|$ for all $x \in A, y \in \mathcal{A}$ [4, Lemma 4, p. 18]. If $A$ is a modular annihilator $A^*$-algebra then $|\cdot|$ is unique [1, (1.3), p. 6] so that $\mathcal{A}$ is also unique.

2. The property (P2). Let $A$ be a Banach algebra, $A^*$ and $A^{**}$ its first and second conjugate spaces. Let $A \hat{\otimes} A^*$ be the projective tensor product of $A$ and $A^*$ [7, pp. 92–95]. Then $A \hat{\otimes} A^*$ is a Banach space with elements of the form $\sum_{k=1}^{\infty} a_k \otimes f_k$ such that $\sum_{k=1}^{\infty} \|a_k\| \|f_k\| < \infty$, $a_k \in A, f_k \in A^*$, and the norm given by

$$\|h\| = \inf \left\{ \sum_{k=1}^{\infty} \|a_k\| \|f_k\| : h = \sum_{k=1}^{\infty} a_k \otimes f_k \right\}.$$ 

For $a \in A, f \in A^*$, let $af \in A^*$ be given by $(af)x = f(xa), x \in A$. This makes $A^*$ into a left Banach $A$-module. We note that if $\sum_{k=1}^{\infty} a_k \otimes f_k \in A \hat{\otimes} A^*$, then $\sum_{k=1}^{\infty} a_k f_k \in A^*$. Let $\psi$ be the continuous linear map of $A \hat{\otimes} A^*$ into $A^*$ given by

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\[ \psi(a \otimes f) = af \quad (a \in A, f \in A^*). \]

For \( a \in A, F \in A^{**} \), let \( F_a \in A^{**} \) be given by \((F_a)f = F(af), f \in A^*\). Then \( A^{**} \) is a right Banach \( A \)-module. (See [6] for the definition and properties of Banach \( A \)-modules.)

Let \( A \circ A^* \) be the Banach space \( A \otimes A^*/\ker(\psi) \) with the usual quotient norm, where \( \ker(\psi) \) is the kernel of \( \psi \). Then \((A \circ A^*)^* \) consists of all those \( \mathcal{F} \in (A \otimes A^*)^* \) which vanish on \( \ker(\psi) \). Now, for each \( F \in A^{**} \), let \( \mathcal{F}_F \in (A \otimes A^*)^* \) be given by
\[
\mathcal{F}_F(a \otimes f) = F(\psi(a \otimes f)) = F(af) \quad (a \in A, f \in A^*). 
\]

Each \( \mathcal{F}_F \) vanishes on \( \ker(\psi) \), so that \( \{\mathcal{F}_F: F \in A^{**}\} \) may be identified as a subspace of \((A \circ A^*)^* \). Moreover if \( Fa = 0 \) for all \( a \in A \) implies \( F = 0 \), then \( F \rightarrow \mathcal{F}_F \) is a one-to-one map of \( A^{**} \) into \((A \otimes A^*)^* \). Thus in this case \( F \rightarrow \mathcal{F}_F \) identifies \( A^{**} \) as a subspace of \((A \circ A^*)^* \).

Let \( \sigma \) denote the \( w^* \)-topology of \((A \circ A^*)^* \).

**Lemma 2.1.** \((A \circ A^*)^* \) is the \( \sigma \)-closure of \( \{\mathcal{F}_F: F \in A^{**}\} \).

**Proof.** We have
\[
\ker(\psi) = \left\{ \sum_{k=1}^{\infty} a_k \otimes f_k \in A \otimes A^*: \sum_{k=1}^{\infty} a_k f_k = 0 \right\} = \left\{ \sum_{k=1}^{\infty} a_k \otimes f_k \in A \otimes A^*: F\left( \sum_{k=1}^{\infty} a_k f_k \right) = 0, F \in A^{**} \right\}.
\]

Thus \( \ker(\psi) = \cap \{\ker(\mathcal{F}_F): F \in A^{**}\} \), which means that \( \ker(\psi) \) is the polar of \( \{\mathcal{F}_F: F \in A^{**}\} \). Therefore, by the Bipolar Theorem [7, p. 126], \((A \circ A^*)^* \) is the \( \sigma \)-closure of \( \{\mathcal{F}_F: F \in A^{**}\} \). This completes the proof.

We observe that if \( A^2 = (0) \), then \( \ker(\mathcal{F}_F) = A \otimes A^* \), for every \( F \in A^{**} \), so that \( \ker(\psi) = A \otimes A^* \) and consequently \((A \circ A^*)^* = (0) \).

Let \( \mathcal{B}(A, A^{**}) \) be the Banach space of all bounded linear operators \( T: A \rightarrow A^{**} \) normed with the operator bound norm. For each \( \mathcal{F} \in (A \otimes A^*)^* \), let \( T_\mathcal{F} \) be the map on \( A \) into \( A^{**} \) given by
\[
(f, T_\mathcal{F}(a)) = \mathcal{F}(a \otimes f) \quad (a \in A, f \in A^*). 
\]

Then clearly \( T_\mathcal{F} \in \mathcal{B}(A, A^{**}) \) for every \( \mathcal{F} \in (A \otimes A^*)^* \), and it is easy to check that the map \( \phi: \mathcal{F} \rightarrow T_\mathcal{F} \) is an isometric isomorphism of \((A \otimes A^*)^* \) onto \( \mathcal{B}(A, A^{**}) \). Give \( \mathcal{B}(A, A^{**}) \) the image of the \( \sigma \)-topology by the map \( \phi \). Following Máté [3], we shall refer to this topology as the ultraweak topology on \( \mathcal{B}(A, A^{**}) \).

Now consider \( A^{**} \) as a right Banach \( A \)-module and let \( \text{Hom}_A(A, A^{**}) \) be the set of all \( T \in \mathcal{B}(A, A^{**}) \) such that \( T(ab) = T(a)b, a, b \in A \). The canonical map \( \pi: A \rightarrow A^{**} \) belongs to \( \mathcal{B}(A, A^{**}) \) since \( \pi(ab)f = f(ab) = \pi(a)(bf) \) for all \( a, b \in A \) and \( f \in A^* \). For each \( F \in A^{**} \), let \( T_F: A \rightarrow A^{**} \) be given by \( T_F(a) = Fa, a \in A \). Then \( T_F \in \text{Hom}_A(A, A^{**}) \), and we have \( \phi(\mathcal{F}_F) \)
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In view of Lemma 2.1 and the fact that the ultraweak closure of \( \{T_F; F \in A^{**}\} \subseteq \text{Hom}_A(A, A^{**}) \) we have

**Lemma 2.2.** \( \phi((A \circ A^*)^*) \subseteq \text{Hom}_A(A, A^{**}) \) and is the ultraweak closure of \( \{T_F; F \in A^{**}\} \).

If \( \phi \) maps \( (A \circ A^*)^* \) onto \( \text{Hom}_A(A, A^{**}) \), we shall write \( (A \circ A^*)^* \cong \text{Hom}_A(A, A^{**}) \). In this case, for every \( T \in \text{Hom}_A(A, A^{**}) \), \( \widetilde{T}_F \in (A \otimes A^*)^* \), given by \( \widetilde{T}_F(a \otimes f) = (f, T(a)) \), belongs to \( (A \circ A^*)^* \). In particular,

\[
\widetilde{T}_a(a \otimes f) = f(a), \quad \text{for all } a, f \in A^*.
\]

We recall that a Banach algebra \( A \) is said to have property (P2) if:

\( a_k \in A, f_k \in A^*, \sum_{k=1}^{\infty} a_k \otimes f_k \in \ker(\psi) \Rightarrow \sum_{k=1}^{\infty} a_k f_k = 0 \) implies that \( \sum_{k=1}^{\infty} f_k(a_k) = 0 \). (This is the left-hand version of the definition given in [8].)

**Theorem 2.3.** Let \( A \) be a Banach algebra. Then the following statements are equivalent:

(i) \( A \) has property (P2).

(ii) For \( h = \sum_{k=1}^{\infty} a_k \otimes f_k \in \ker(\psi) \) we have \( \sum_{k=1}^{\infty} f_k(a_k) = 0 \).

(iii) \( \widetilde{T}_a \) vanishes on \( \ker(\psi) \).

(iv) \( \widetilde{T}_a \in (A \circ A^*)^* \).

(v) \( \text{Hom}_A(A, A^{**}) \cong (A \circ A^*)^* \).

(vi) There exists a net \( \{u_a\} \) in \( A \) such that \( \{\widetilde{T}_{u_a}\} \) converges to \( \widetilde{T}_a \) in the \( \sigma \)-topology on \( (A \otimes A^*)^* \).

**Proof.** (i) \( \Leftrightarrow \) (ii) and (iii) \( \Leftrightarrow \) (iv) are clear.

(iv) \( \Rightarrow \) (vi). Suppose (iv) holds. Then \( \ker(\widetilde{T}_a) \supset \ker(\psi) \). We have

\[
\widetilde{T}_a(h) = \widetilde{T}_a \left( \sum_{k=1}^{\infty} a_k \otimes f_k \right) = \sum_{k=1}^{\infty} \widetilde{T}_a(a_k \otimes f_k) = \sum_{k=1}^{\infty} f_k(a_k),
\]

for all \( h = \sum_{k=1}^{\infty} a_k \otimes f_k \in A \otimes A^* \). Since \( \{\widetilde{T}_F; F \in A^{**}\} \) is \( \sigma \)-dense in \( (A \circ A^*)^* \), there exists a net \( \{F_n\} \) in \( A^{**} \) such that \( \widetilde{T}_{F_n}(h) \to \widetilde{T}_a(h) \) for all \( h \in A \otimes A^* \). Since \( \sigma(A) \) is \( \sigma \)-dense in \( A^{**} \) and \( \sigma \) is weaker than the \( \sigma \)-topology on \( A^{**} \), it follows that \( \{\widetilde{T}_{F_n}; a \in A\} \) is \( \sigma \)-dense in \( (A \circ A^*)^* \). Hence there exists a net \( \{u_a\} \) in \( A \) such that \( \{\widetilde{T}_{u_a}\} \sigma \)-converges to \( \widetilde{T}_a \). We have

\[
\widetilde{T}_{u_a}(h) = \pi(u_a)(\psi(h)) = \sum_{k=1}^{\infty} \pi(u_a)(a_k f_k)
\]

\[
= \sum_{k=1}^{\infty} a_k f_k(u_a) = \sum_{k=1}^{\infty} (a_k f_k)(u_a)
\]

\[
= \sum_{k=1}^{\infty} f_k(u_a a_k).
\]

Thus
for all \( h = \sum_{k=1}^{\infty} a_k \otimes f_k \in A \otimes A^* \).

(vi) \( \Rightarrow \) (v). We have \( \phi((A \circ A^*)^*) \subseteq \text{Hom}_A(A, A^{**}) \). We need only show that \( \text{Hom}_A(A, A^{**}) \subseteq \phi((A \circ A^*)^*) \). Let \( T \in \text{Hom}_A(A, A^{**}) \) and let \( \mathcal{F}_T \) be the corresponding element of \( (A \otimes A^*)^* \). Then, using (1), we obtain (identifying \( A \) as a subset of \( A^{**} \) and \( A^* \) as a subset of \( A^{***} \)):

\[
\mathcal{F}_T \left( \sum_{k=1}^{\infty} a_k \otimes f_k \right) = \sum_{k=1}^{\infty} \mathcal{F}_T (a_k \otimes f_k) = \sum_{k=1}^{\infty} (f_k(T(a_k)))
\]

\[
= \sum_{k=1}^{\infty} (T^*f_k)(a_k) = \lim_{\alpha} \sum_{k=1}^{\infty} (T^*f_k)(u_\alpha a_k)
\]

\[
= \lim_{\alpha} \sum_{k=1}^{\infty} f_k(T(u_\alpha a_k)) = \lim_{\alpha} \sum_{k=1}^{\infty} f_k(T(u_\alpha)a_k)
\]

\[
= \lim_{\alpha} \sum_{k=1}^{\infty} (T^*a_\alpha f_k)(u_\alpha) = \sum_{k=1}^{\infty} T^*(a_\alpha f_k)(u_\alpha)
\]

where \( T^* \) is the conjugate of \( T \). Hence if \( \sum_{k=1}^{\infty} a_\alpha f_k = 0 \) then \( \mathcal{F}_T(\sum_{k=1}^{\infty} a_k \otimes f_k) = 0 \), so that \( \ker(\psi) \subseteq \ker(\mathcal{F}_T) \). Thus \( \text{Hom}_A(A, A^{**}) \subseteq \phi((A \circ A^*)^*) \) and so \( \text{Hom}_A(A, A^{**}) \cong (A \circ A^*)^* \).

(v) \( \Rightarrow \) (iv). This is clear since \( \sigma \in \text{Hom}_A(A, A^{**}) \).

3. Dual \( A^* \)-algebras. Let \( A \) be a Banach algebra. A map \( T: A \to A \) is called a left (resp. right) multiplier if \( T(ab) = T(a)b \) (resp. \( T(ab) = aT(b) \)), for all \( a, b \in A \). Let \( M_l(A) \) (resp. \( M_r(A) \)) be the set of all bounded linear left (resp. right) multipliers on \( A \). \( M_l(A) \) and \( M_r(A) \) are Banach algebras under the usual operations for operators and the operator bound norm. We observe that if \( T \in M_l(A) \) then the composite map \( \pi_1 \circ T \in \text{Hom}_A(A, A^{**}) \). Let \( \phi_\pi \) be the map of \( M_l(A) \) into \( \text{Hom}_A(A, A^{**}) \) given by

\[
\phi_\pi(T) = \pi \circ T \quad (T \in M_l(A)).
\]

For any Banach space \( X \), let \( S(X) \) denote the closed unit ball of \( X \). It follows from the proof of [8, Theorem 4.7, p. 286] that if \( A \) is a dual \( A^* \)-algebra of the first kind then \( S(M_l(A)) \) is \( \tau_\pi \)-compact, where \( \tau_\pi \) is the weak operator topology on \( M_l(A) \). (We take the left-hand version of the arguments in [8, p. 286].)

**Theorem 3.1.** Let \( A \) be an \( A^* \)-algebra of the first kind. Then the following statements are equivalent:

(i) \( A \) is dual.

(ii) \( \phi_\pi(M_l(A)) \) is the ultraweak closure of \( \{ TF : F \in A^{**} \} \).

**Proof.** (i) \( \Rightarrow \) (ii). Suppose \( A \) is dual. Then, by [8, Theorem 4.7, p. 286], it has property (P2) and therefore, by Theorem 2.3, \( (A \circ A^*)^* \cong \text{Hom}_A(A, A^{**}) \). Hence \( \text{Hom}_A(A, A^{**}) \) is the ultraweak closure of \( \{ TF : F \in \)
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Now $T_F(a)f = (Fa)f = (F \circ \pi(a))f$ and, by [9, Theorem 5.2, p. 830],
$\pi(A)$ is an ideal of $A^{**}$ when $A^{**}$ is given either Arens product, so that
$T_F(a) \in \pi(A)$ for all $a \in A$. Therefore $T_F = \pi \circ T$, for some $T \in M_f(A)$,
and so $\{T_F : F \in A^{**}\} \subseteq \phi_\pi(M_f(A))$. Let $Q \in \text{Hom}_A(A, A^{**})$.
Then, by Lemma 2.2 and the fact that $(A \circ A^\ast)^* = \text{Hom}_A(A, A^{**})$, there exists a net
$(F_\alpha)$ in $A^{**}$ such that $(f, T_{F_\alpha}(a)) \to (f, Q(a))$ for all $a \in A, f \in A^\ast$. Let
$T_\alpha \in M_f(A)$ be such that $T_{F_\alpha} = \pi \circ T_\alpha$, for all $\alpha$. Then $(f, T_{F_\alpha}(a)) = (f(T_\alpha(a)))$
since $T_\alpha(a) \in A$. But, by [8, Theorem 4.7, p. 286], $M_f(A)$ is $\tau_f$-complete.
Hence there exists $T \in M_f(A)$ such that $f(T_\alpha(a)) \to f(T(a))$, for all $a \in A, f \in A^\ast$. This shows that
$\pi(T(a))f = (f, Q(a))$, for all $a \in A, f \in A^\ast$, or equivalently, $\pi(T(a)) = Q(a)$, for all $a \in A$, i.e., $Q = \pi \circ T$. Thus $Q \in \phi_\pi(M_f(A))$ and so $\text{Hom}_A(A, A^{**}) = \phi_\pi(M_f(A))$. Since $(A \circ A^\ast)^* = \text{Hom}_A(A, A^{**})$ and since $(A \circ A^\ast)^*$ is the $\sigma$-closure of $\{F_F : F \in A^{**}\}$, it follows that $\phi_\pi(M_f(A))$ is the ultraweak closure of $\{T_F : F \in A^{**}\}$.

(ii) $\Rightarrow$ (i). Suppose (ii) holds. Then, in view of Lemma 2.2, $M_f(A)$ is
isometrically isomorphic to $(A \circ A^\ast)^\ast$. From Lemma 2.1 we obtain
$\mathfrak{S}((A \circ A^\ast)^\ast)$ is $\sigma$-compact, and since $\tau_f$ is weaker than the ultraweak topology
on $M_f(A)$, it follows that $\mathfrak{S}(M_f(A))$ is $\tau_f$-compact and therefore $\tau_f$-complete. Let $I$ be the identity element of $M_f(A)$. Since $\{T_F : F \in A^{**}\}$ is
ultraweak dense in $\phi_\pi(M_f(A))$, there exists a net $(F_\alpha)$ in $A^{**}$ such that $T_{F_\alpha}$
converges ultraweakly to $\pi \circ I = \pi$, or equivalently, $\mathfrak{S}_{F_\alpha}$ $\sigma$-converges to $\mathfrak{S}_\pi$.
Since $\pi(A)$ is $w^\ast$-dense in $A^{**}$ and the $w^\ast$-topology is stronger than the
$\sigma$-topology on $A^{**}$, it follows that there exists a net $(u_\alpha)$ in $A$ such that
$(\mathfrak{S}_{\pi(u_\alpha)})$ $\sigma$-converges to $\mathfrak{S}_\pi$ and so, by Theorem 2.3, $A$ has property (P2).
Therefore, by [8, Theorem 4.7, p. 287], $A$ is dual.

COROLLARY 3.2. Let $A$ be an $A^\ast$-algebra of the first kind. Then $A$ is dual if
and only if $\phi_\pi(M_f(A)) = \phi((A \circ A^\ast)^\ast)$.

COROLLARY 3.3. Let $A$ be a modular annihilator $A^\ast$-algebra of the first kind.
If $\text{Hom}_A(A, A^{**})$ is the ultraweak closure of $\{T_F : F \in A^{**}\}$ then $A$ is dual.

Proof. By [9, Theorem 5.2, p. 830], $\pi(A)$ is an ideal of $A^{**}$ so that $T_F$ maps
$A$ into $\pi(A)$ for every $F \in A^{**}$. Hence if $\text{Hom}_A(A, A^{**})$ is the ultraweak closure of $\{T_F : F \in A^{**}\}$, then $\text{Hom}_A(A, A^{**}) = \phi_\pi(M_f(A))$ by the proof
above. Therefore $A$ is dual by Theorem 3.1.

4. A realization of the algebra $M_f(A)$.

THEOREM 4.1. Let $A$ be a dual $A^\ast$-algebra of the first kind and let $\mathfrak{A}$ be its
completion. Let $\pi_\mathfrak{A}$ be the canonical map of $\mathfrak{A}$ into $\mathfrak{A}^{**}$. Then $\pi_\mathfrak{A}(A)$ is an ideal
of $\mathfrak{A}^{**}$ when $\mathfrak{A}^{**}$ is given Arens product.

Proof. Let $x \in A$, $F \in \mathfrak{A}^{**}$ and let $(\{e_\alpha\}$ be a maximal orthogonal family
of selfadjoint minimal idempotents in $A$. By [4, Theorem 16, p. 30], $\Sigma_\alpha e_\alpha x$ is
summable to $x$ in the norm $\| \cdot \|$, and hence there exists only a countable
number of $e_\alpha$ for which $e_\alpha x \neq 0$, say $e_{\alpha_1}, e_{\alpha_2}, \ldots$. Since $A$ and $\mathfrak{A}$ have the
same socle and \( \pi_\mathfrak{A}(\mathfrak{A}) \) is an ideal of \( \mathfrak{A}^{**} \), it follows that \( F \ast \pi_\mathfrak{A}(e_{\alpha_i}) \in \pi_\mathfrak{A}(A) \) for \( i = 1, 2, \ldots \). Let \( m, n \) be positive integers, \( m < n \). By [4, Lemma 4, p. 18], we have

\[
\left\| \sum_{i=1}^{n} F \ast \pi_\mathfrak{A}(e_{\alpha_i}) \pi_\mathfrak{A}(x) - \sum_{i=1}^{m} F \ast \pi_\mathfrak{A}(e_{\alpha_i}) \ast \pi_\mathfrak{A}(x) \right\|
\]

\[
= \left\| \left( \sum_{i=m+1}^{n} F \ast \pi_\mathfrak{A}(e_{\alpha_i}) \right) \left( \sum_{i=m+1}^{n} \pi_\mathfrak{A}(e_{\alpha_i}) \ast \pi_\mathfrak{A}(x) \right) \right\|
\]

\[
\leq k \left\| \left( \sum_{i=m+1}^{n} F \ast \pi_\mathfrak{A}(e_{\alpha_i}) \right) \left( \sum_{i=m+1}^{n} \pi_\mathfrak{A}(e_{\alpha_i}) \ast \pi_\mathfrak{A}(x) \right) \right\|
\]

\[
\leq k |F| \left\| \sum_{i=m+1}^{n} \pi_\mathfrak{A}(e_{\alpha_i}) \ast \pi_\mathfrak{A}(x) \right\|
\]

where \( |F| \) denotes the norm of \( F \) in \( \mathfrak{A}^{**} \) and \( k \) is a positive constant. Thus \( \{ \sum_{i=1}^{n} F \ast \pi_\mathfrak{A}(e_{\alpha_i}) \ast \pi_\mathfrak{A}(x) \} \) is a Cauchy sequence in \( \pi_\mathfrak{A}(A) \) with respect to the norm \( \| \cdot \| \), and so there exists \( z \in A \) such that \( \pi_\mathfrak{A}(z) = \sum_{i=1}^{\infty} F \ast \pi_\mathfrak{A}(e_{\alpha_i}) \ast \pi_\mathfrak{A}(x) \). Since \( \sum_{i=1}^{\infty} F \ast \pi_\mathfrak{A}(e_{\alpha_i}) \ast \pi_\mathfrak{A}(x) \) also converges to \( \pi_\mathfrak{A}(z) \) and to \( F \ast \pi_\mathfrak{A}(x) \) in the norm \( \| \cdot \| \), we have \( \pi_\mathfrak{A}(z) = F \ast \pi_\mathfrak{A}(x) \). Hence \( F \ast \pi_\mathfrak{A}(x) \in \pi_\mathfrak{A}(A) \), for all \( x \in A \) and \( F \in \mathfrak{A}^{**} \). Similarly we can show that \( \pi_\mathfrak{A}(x) \ast F \in \mathfrak{A}^{**} \), for all \( x \in A \) and \( F \in \mathfrak{A}^{**} \). Therefore \( \pi_\mathfrak{A}(A) \) is an ideal of \( \mathfrak{A}^{**} \).

**Theorem 4.2.** Let \( A \) be a dual \( A^* \)-algebra of the first kind and \( \mathfrak{A} \) its completion. Then \( M_{\mathfrak{A}}(A) \) is algebra isomorphic to \( \mathfrak{A}^{**} \) when \( \mathfrak{A}^{**} \) is given Arens product. This isomorphism is given by the following relation: For each \( T \in M_{\mathfrak{A}}(A) \) there exists a unique \( F_T \in \mathfrak{A}^{**} \) such that

\[ \pi_\mathfrak{A}(Tx) = F_T \ast \pi_\mathfrak{A}(x) \quad (x \in A). \]

**Proof.** For each \( x \in A \), let \( \|x\|_A' = \sup\{\|xy\| : \|y\| < 1, y \in A\} \). Then \( \| \cdot \|_A' \) is a norm on \( A \) which is equivalent to \( \| \cdot \| \) [4, Theorem 18, p. 31]. Hence if \( T \in M_{\mathfrak{A}}(A) \) and \( x \in A \), then

\[ \|Tx\|_A' = \sup\{\|T(x)y\| : \|y\| < 1, y \in A\} \]

\[ = \sup\{\|T(xy)\| : \|y\| < 1, y \in A\} \]

\[ \leq \|T\|\sup\{\|xy\| : \|y\| < 1, y \in A\} \]

\[ \leq k'\|T\|\|x\|. \]

where \( k' \) is a constant \( > 0 \). Thus \( |Tx| \leq k''|x| \) for all \( x \in A \) and some constant \( k'' > 0 \). Since \( A \) is dense in \( \mathfrak{A} \), it follows that \( T \) has a unique bounded extension \( T' \) to \( \mathfrak{A} \). Clearly \( T' \in M_{\mathfrak{A}}(\mathfrak{A}) \). By [2, Corollary 3.2, p. 509], there exists a unique \( F_T \in \mathfrak{A}^{**} \) such that \( \pi_\mathfrak{A}(Tx) = F_T \ast \pi_\mathfrak{A}(x) \) for all \( x \in A \).
Since, by Theorem 4.1, \( \pi_R(A) \) is an ideal of \( \mathfrak{H}^{**} \), we have that \( T \to F_T \) is an algebra isomorphism of \( M_f(A) \) onto \( \mathfrak{H}^{**} \).

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