MULTIPLIERS ON DUAL \( A^*\)-ALGEBRAS

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Abstract. Let \( A \) be an \( \mathcal{A}^*\)-algebra which is a dense \( *\)-ideal of a \( B^*\)-algebra \( \mathcal{A} \). We use tensor products and the algebra \( M_\ell(A) \) of left multipliers on \( A \) to obtain a characterization of duality in \( A \). We show, moreover, that if \( A \) is dual then \( M_\ell(A) \) is algebra isomorphic to the second conjugate space \( \mathcal{A}^{**} \) of \( \mathcal{A} \) when \( \mathcal{A}^{**} \) is given Arens product.

1. Introduction. Let \( A \) be an \( \mathcal{A}^*\)-algebra which is a dense \( *\)-ideal of a \( B^*\)-algebra \( \mathcal{A} \). In [8] a necessary and sufficient condition was given for \( A \) to be dual which was expressed in terms of the weak operator topology on \( M_\ell(A) \), the algebra of right multipliers on \( A \), and a certain property of \( A \) called property (P2). In this paper we give several characterizations of property (P2) and then use some of them to give conditions for duality in \( A \).

Our presentation here differs somewhat from that in [8]. We use the tensor product approach as developed in [3] and [6]. Particularly in §2 we follow closely the presentation given in [3].

We shall use the notation of [8]. An \( \mathcal{A}^*\)-algebra \( \mathcal{A} \) is said to be of the first kind if it is an ideal of its completion \( \mathcal{H} \) in the auxiliary norm \( | \cdot | \). It follows that there exists a constant \( k > 0 \) such that \( |xy| \leq k|x||y| \) for all \( x \in \mathcal{A}, y \in \mathcal{A} \) [4, Lemma 4, p. 18]. If \( A \) is a modular annihilator \( \mathcal{A}^*\)-algebra then \( | \cdot | \) is unique [1, (1.3), p. 6] so that \( \mathcal{H} \) is also unique.

2. The property (P2). Let \( A \) be a Banach algebra, \( A^* \) and \( A^{**} \) its first and second conjugate spaces. Let \( A \hat{\otimes} A^* \) be the projective tensor product of \( A \) and \( A^* \) [7, pp. 92–95]. Then \( A \hat{\otimes} A^* \) is a Banach space with elements of the form \( \sum_{k=1}^{\infty} a_k \hat{\otimes} f_k \) such that \( \sum_{k=1}^{\infty} \|a_k\| \|f_k\| < \infty \), \( a_k \in A \), \( f_k \in A^* \), and the norm given by

\[
\|h\| = \inf \left\{ \sum_{k=1}^{\infty} \|a_k\| \|f_k\| : h = \sum_{k=1}^{\infty} a_k \hat{\otimes} f_k \right\}.
\]

For \( a \in A, f \in A^* \), let \( af \in A^* \) be given by \( (af)x = f(xa), x \in A \). This makes \( A^* \) into a left Banach \( A \)-module. We note that if \( \sum_{k=1}^{\infty} a_k \hat{\otimes} f_k \in A \hat{\otimes} A^* \), then \( \sum_{k=1}^{\infty} a_k f_k \in A^* \). Let \( \psi \) be the continuous linear map of \( A \hat{\otimes} A^* \) into \( A^* \) given by

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For \( a \in A, f \in A^* \), let \( Fa \in A^{**} \) be given by \( (Fa)f = F(af), f \in A^* \). Then \( A^{**} \) is a right Banach \( A \)-module. (See [6] for the definition and properties of Banach \( A \)-modules.)

Let \( A \circ A^* \) be the Banach space \( A \hat{\otimes} A^*/\ker(\psi) \) with the usual quotient norm, where \( \ker(\psi) \) is the kernel of \( \psi \). Then \( (A \circ A^*)^* \) consists of all those \( \mathcal{F} \in (A \hat{\otimes} A^*)^* \) which vanish on \( \ker(\psi) \). Now, for each \( F \in A^{**} \), let \( \mathcal{F}_F \in (A \hat{\otimes} A^*)^* \) be given by

\[
\mathcal{F}_F(a \hat{\otimes} f) = F(\psi(a \hat{\otimes} f)) = F(af) \quad (a \in A, f \in A^*).
\]

Each \( \mathcal{F}_F \) vanishes on \( \ker(\psi) \), so that \( \{ \mathcal{F}_F : F \in A^{**} \} \) may be identified as a subspace of \( (A \circ A^*)^* \). Moreover if \( Fa = 0 \) for all \( a \in A \) implies \( F = 0 \), then \( F \to \mathcal{F}_F \) is a one-to-one map of \( A^{**} \) into \( (A \hat{\otimes} A^*)^* \). Thus in this case \( F \to \mathcal{F}_F \) identifies \( A^{**} \) as a subspace of \( (A \circ A^*)^* \).

Let \( \sigma \) denote the \( w^* \)-topology of \( (A \hat{\otimes} A^*)^* \).

**Lemma 2.1.** \( (A \circ A^*)^* \) is the \( \sigma \)-closure of \( \{ \mathcal{F}_F : F \in A^{**} \} \).

**Proof.** We have

\[
\ker(\psi) = \left\{ \sum_{k=1}^\infty a_k \hat{\otimes} f_k \in A \hat{\otimes} A^* : \sum_{k=1}^\infty a_k f_k = 0 \right\}
\]

Thus \( \ker(\psi) = \cap \ker(\mathcal{F}_F) : F \in A^{**} \), which means that \( \ker(\psi) \) is the polar of \( \{ \mathcal{F}_F : F \in A^{**} \} \). Therefore, by the Bipolar Theorem [7, p. 126], \( (A \circ A^*)^* \) is the \( \sigma \)-closure of \( \{ \mathcal{F}_F : F \in A^{**} \} \). This completes the proof.

We observe that if \( A^2 = (0) \), then \( \ker(\mathcal{F}_F) = A \hat{\otimes} A^* \), for every \( F \in A^{**} \), so that \( \ker(\psi) = A \hat{\otimes} A^* \) and consequently \( (A \circ A^*)^* = (0) \).

Let \( \mathcal{B}(A, A^{**}) \) be the Banach space of all bounded linear operators \( T: A \to A^{**} \) normed with the operator bound norm. For each \( \mathcal{F} \in (A \hat{\otimes} A^*)^* \), let \( T_\mathcal{F} \) be the map on \( A \) into \( A^{**} \) given by

\[
(f, T_\mathcal{F}(a)) = \mathcal{F}(a \hat{\otimes} f) \quad (a \in A, f \in A^*).
\]

Then clearly \( T_\mathcal{F} \in \mathcal{B}(A, A^{**}) \) for every \( \mathcal{F} \in (A \hat{\otimes} A^*)^* \), and it is easy to check that the map \( \phi: \mathcal{F} \to T_\mathcal{F} \) is an isometric isomorphism of \( (A \hat{\otimes} A^*)^* \) onto \( \mathcal{B}(A, A^{**}) \). Give \( \mathcal{B}(A, A^{**}) \) the image of the \( \sigma \) topology by the map \( \phi \).

Now consider \( A^{**} \) as a right Banach \( A \)-module and let \( \text{Hom}_A(A, A^{**}) \) be the set of all \( T \in \mathcal{B}(A, A^{**}) \) such that \( T(ab) = T(a)b, a, b \in A \). The canonical map \( \pi: A \to A^{**} \) belongs to \( \mathcal{B}(A, A^{**}) \) since \( \pi(ab)f = f(ab) = \pi(a)(bf) \) for all \( a, b \in A \) and \( f \in A^* \). For each \( F \in A^{**} \), let \( T_F: A \to A^{**} \) be given by \( T_F(a) = Fa, a \in A \). Then \( T_F \in \text{Hom}_A(A, A^{**}) \), and we have \( \phi(\mathcal{F}_F) \)
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$= T_F$ for all $F \in A^{**}$. In view of Lemma 2.1 and the fact that the ultraweak closure of $\{T_F: F \in A^{**}\} \subseteq \text{Hom}_A(A, A^{**})$ we have

**Lemma 2.2.** $\phi((A \circ A^*)^*) \subseteq \text{Hom}_A(A, A^{**})$ and is the ultraweak closure of $\{T_F: F \in A^{**}\}$.

If $\phi$ maps $(A \circ A^*)^*$ onto $\text{Hom}_A(A, A^{**})$, we shall write $(A \circ A^*)^* = \text{Hom}_A(A, A^{**})$. In this case, for every $T \in \text{Hom}_A(A, A^{**})$, $\mathcal{F}_T \in (A \hat{\otimes} A^*)^*$, given by $\mathcal{F}_T(a \hat{\otimes} f) = (f, T(a))$, belongs to $(A \circ A^*)^*$. In particular,

$$\mathcal{F}_T(a \hat{\otimes} f) = f(a), \quad \text{for all } a, f \in A^*.$$

We recall that a Banach algebra $A$ is said to have property (P2) if:

- $a_k \in A, f_k \in A^*$, $\sum_{k=1}^{\infty} \|a_k\| \|f_k\| < \infty$ and $\sum_{k=1}^{\infty} a_k f_k = 0$ implies that $\sum_{k=1}^{\infty} f_k(a_k) = 0$. (This is the left-hand version of the definition given in [8].)

**Theorem 2.3.** Let $A$ be a Banach algebra. Then the following statements are equivalent:

(i) $A$ has property (P2).
(ii) For $h = \sum_{k=1}^{\infty} a_k \hat{\otimes} f_k \in \ker(\psi)$ we have $\sum_{k=1}^{\infty} f_k(a_k) = 0$.
(iii) $\mathcal{F}_\pi$ vanishes on $\ker(\psi)$.
(iv) $\mathcal{F}_\pi \in (A \circ A^*)^*$.
(v) $\text{Hom}_A(A, A^{**}) = (A \circ A^*)^*$.
(vi) There exists a net $\{u_a\}$ in $A$ such that $\mathcal{F}_{\pi(u_a)}$ converges to $\mathcal{F}_\pi$ in the w*-topology on $(A \hat{\otimes} A^*)^*$.

**Proof.** (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv) are clear.

(iv) $\Rightarrow$ (vi). Suppose (iv) holds. Then $\ker(\mathcal{F}_\pi) \supset \ker(\psi)$. We have

$$\mathcal{F}_\pi(h) = \mathcal{F}_\pi\left(\sum_{k=1}^{\infty} a_k \hat{\otimes} f_k\right) = \sum_{k=1}^{\infty} \mathcal{F}_\pi(a_k \hat{\otimes} f_k) = \sum_{k=1}^{\infty} f_k(a_k),$$

for all $h = \sum_{k=1}^{\infty} a_k \hat{\otimes} f_k \in A \hat{\otimes} A^*$. Since $\{\mathcal{F}_F: F \in A^{**}\}$ is $\sigma$-dense in $(A \circ A^*)^*$, there exists a net $\{F_\alpha\}$ in $A^{**}$ such that $\mathcal{F}_{F_\alpha}(h) \to \mathcal{F}_\pi(h)$ for all $h \in A \hat{\otimes} A^*$. Since $\pi(A)$ is w*-dense in $A^{**}$ and $\sigma$ is weaker than the w*-topology on $A^{**}$, it follows that $\{\mathcal{F}_{\pi(a)}: a \in A\}$ is $\sigma$-dense in $(A \circ A^*)^*$. Hence there exists a net $\{u_a\}$ in $A$ such that $(\mathcal{F}_{\pi(u_a)})$ $\sigma$-converges to $\mathcal{F}_\pi$. We have

$$\mathcal{F}_{\pi(u_a)}(h) = \pi(u_a)(\psi(h)) = \sum_{k=1}^{\infty} \pi(u_a)(a_k f_k) = \sum_{k=1}^{\infty} a_k f_k(u_a) = \sum_{k=1}^{\infty} f_k(u_a a_k).$$

Thus
\[
\lim_{a} \mathcal{S}_{\psi(u_{a})}(h) = \lim_{a} \sum_{k=1}^{\infty} f_{k}(u_{a}a_{k}) = \sum_{k=1}^{\infty} f_{k}(a_{k}) = \mathcal{S}_{\psi}(h),
\]
for all \( h = \sum_{k=1}^{\infty} a_{k} \otimes f_{k} \in A \otimes A^{*} \).

(vi) \( \Rightarrow \) (v). We have \( \phi((A \circ A^{*})^{*}) \subseteq \text{Hom}_{A}(A, A^{**}) \). We need only show that \( \text{Hom}_{A}(A, A^{**}) \subseteq \phi((A \circ A^{*})^{*}) \). Let \( T \in \text{Hom}_{A}(A, A^{**}) \) and let \( \mathcal{S}_{T} \) be the corresponding element of \( (A \otimes A^{*})^{*} \). Then, using (1), we obtain (identifying \( A \) as a subset of \( A^{**} \) and \( A^{*} \) as a subset of \( A^{***} \)):

\[
\mathcal{S}_{T} \left( \sum_{k=1}^{\infty} a_{k} \otimes f_{k} \right) = \sum_{k=1}^{\infty} \mathcal{S}_{T}(a_{k} \otimes f_{k}) = \sum_{k=1}^{\infty} (f_{k}(T(a_{k})))
= \lim_{a} \sum_{k=1}^{\infty} (T^{*}f_{k})(a_{k}) = \lim_{a} \sum_{k=1}^{\infty} (T^{*}f_{k})(a_{k})
= \lim_{a} \sum_{k=1}^{\infty} (a_{k}T^{*}f_{k})(u_{a}) = \sum_{k=1}^{\infty} (T^{*}f_{k})(u_{a})
= \lim_{a} \sum_{k=1}^{\infty} (a_{k}T^{*}f_{k})(u_{a}) = \sum_{k=1}^{\infty} (T^{*}f_{k})(u_{a})
\]
where \( T^{*} \) is the conjugate of \( T \). Hence if \( \sum_{k=1}^{\infty} a_{k}f_{k} = 0 \) then \( \mathcal{S}_{T}(\sum_{k=1}^{\infty} a_{k} \otimes f_{k}) = 0 \), so that \( \ker(\psi) \subseteq \ker(\mathcal{S}_{T}) \). Thus \( \text{Hom}_{A}(A, A^{**}) \subseteq \phi((A \circ A^{*})^{*}) \) and so \( \text{Hom}_{A}(A, A^{**}) \cong (A \circ A^{*})^{*} \).

(v) \( \Rightarrow \) (iv). This is clear since \( \sigma \in \text{Hom}_{A}(A, A^{**}) \).

3. Dual \( A^{*} \)-algebras. Let \( A \) be a Banach algebra. A map \( T: A \to A \) is called a left (resp. right) multiplier if \( T(ab) = T(a)b \) (resp. \( T(ab) = aT(b) \)), for all \( a, b \in A \). Let \( M_{l}(A) \) (resp. \( M_{r}(A) \)) be the set of all bounded linear left (resp. right) multipliers on \( A \). \( M_{l}(A) \) and \( M_{r}(A) \) are Banach algebras under the usual operations for operators and the operator bound norm. We observe that if \( T \in M_{l}(A) \) then the composite map \( \pi \circ T \in \text{Hom}_{A}(A, A^{**}) \). Let \( \phi_{\pi} \) be the map of \( M_{l}(A) \) into \( \text{Hom}_{A}(A, A^{**}) \) given by

\[
\phi_{\pi}(T) = \pi \circ T \quad (T \in M_{l}(A)).
\]

For any Banach space \( X \), let \( \mathcal{S}(X) \) denote the closed unit ball of \( X \). It follows from the proof of [8, Theorem 4.7, p. 286] that if \( A \) is a dual \( A^{*} \)-algebra of the first kind then \( \mathcal{S}(M_{l}(A)) \) is \( \tau_{r} \)-compact, where \( \tau_{r} \) is the weak operator topology on \( M_{l}(A) \). (We take the left-hand version of the arguments in [8, p. 286].)

**Theorem 3.1.** Let \( A \) be an \( A^{*} \)-algebra of the first kind. Then the following statements are equivalent:

(i) \( A \) is dual.

(ii) \( \phi_{\pi}(M_{l}(A)) \) is the ultraweak closure of \( \{ T_{F} : F \in A^{**} \} \).

**Proof.** (i) \( \Rightarrow \) (ii). Suppose \( A \) is dual. Then, by [8, Theorem 4.7, p. 286], it has property (P2) and therefore, by Theorem 2.3, \( (A \circ A^{*})^{*} \cong \text{Hom}_{A}(A, A^{**}) \). Hence \( \text{Hom}_{A}(A, A^{**}) \) is the ultraweak closure of \( \{ T_{F} : F \in A^{**} \} \).
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Now $T_F(a)f = (Fa)f = (F \ast \pi(a))f$ and, by [9, Theorem 5.2, p. 830],
$
\pi(A)
$

is an ideal of $A^{**}$ when $A^{**}$ is given either Arens product, so that
$T_F(a) \in \pi(A)$ for all $a \in A$. Therefore $T_F = \pi \circ T$, for some $T \in M_f(A)$,
and so $\{T_F : F \in A^{**}\} \subseteq \phi_f(M_f(A))$. Let $Q \in \text{Hom}_A(A, A^{**})$. Then, by
Lemma 2.2 and the fact that $(A \circ A^*)^* = \text{Hom}_A(A, A^{**})$, there exists a net
$\{F_a\}$ in $A^{**}$ such that $(f, T_{F_a}(a)) \to (f, \pi(a))$ for all $a \in A, f \in A^*$. Let
$T_a \in M_f(A)$ be such that $T_{F_a} = \pi \circ T_a$, for all $a$. Then $(f, T_{F_a}(a)) = (f, T_a(a))$
since $T_a(a) \in A$. But, by [8, Theorem 4.7, p. 286], $M_f(A)$ is $\tau_f$-complete.
Hence there exists $T \in M_f(A)$ such that $f(T_{F_a}(a)) \to f(T(a))$, for all $a \in A, f \in A^*$. This shows that
$\pi(T(a))f = (f, \pi(a))$, for all $a \in A, f \in A^*$, or equivalently, $\pi(T(a)) = Q(a)$, for all $a \in A$, i.e., $Q = \pi \circ T$. Thus $Q \in \phi_f(M_f(A))$ and so $\text{Hom}_A(A, A^{**}) = \phi_f(M_f(A))$. Since $(A \circ A^*)^* \cong \text{Hom}_A(A, A^{**})$ and since $(A \circ A^*)^*$ is the $\sigma$-closure of $\{F : F \in A^{**}\}$, it
follows that $\phi_f(M_f(A))$ is the ultraweak closure of $\{T_F : F \in A^{**}\}$.

$(ii) \Rightarrow (i)$. Suppose $(ii)$ holds. Then, in view of Lemma 2.2, $M_f(A)$ is
isometrically isomorphic to $(A \circ A^*)^*$. From Lemma 2.1 we obtain
$\mathcal{S}(A \circ A^*)$ is $\sigma$-compact, and since $\tau_f$ is weaker than the ultraweak topology
on $M_f(A)$, it follows that $\mathcal{S}(M_f(A))$ is $\tau_f$-compact and therefore $\tau_f$-complete. Let $I$
be the identity element of $M_f(A)$. Since $\{T_F : F \in A^{**}\}$ is ultraweak dense in $\phi_f(M_f(A))$, there exists a net $\{F_a\}$ in $A^{**}$ such that $T_{F_a}$
converges ultraweakly to $\pi \circ I = \pi$, or equivalently, $\mathcal{S}(\mathcal{F}_F)$ $\sigma$-converges to $\mathcal{S}(\pi)$. Since $\pi(A)$ is $w^*$-dense in $A^{**}$ and the $w^*$-topology is stronger than the
$\sigma$-topology on $A^{**}$, it follows that there exists a net $\{u_a\}$ in $A$ such that
$\mathcal{S}(\pi(u_a))$ $\sigma$-converges to $\mathcal{S}(\pi)$ and so, by Theorem 2.3, $A$ has property (P2).
Therefore, by [8, Theorem 4.7, p. 287], $A$ is dual.

**Corollary 3.2.** Let $A$ be an $A^*$-algebra of the first kind. Then $A$ is dual if
and only if $\phi_f(M_f(A)) = \phi((A \circ A^*)^*)$.

**Corollary 3.3.** Let $A$ be a modular annihilator $A^*$-algebra of the first kind.
If $\text{Hom}_A(A, A^{**})$ is the ultraweak closure of $\{T_F : F \in A^{**}\}$ then $A$ is dual.

**Proof.** By [9, Theorem 5.2, p. 830], $\pi(A)$ is an ideal of $A^{**}$ so that $T_F$
maps $A$ into $\pi(A)$ for every $F \in A^{**}$. Hence if $\text{Hom}_A(A, A^{**})$ is the ultraweak closure of $\{T_F : F \in A^{**}\}$, then $\text{Hom}_A(A, A^{**}) = \phi_f(M_f(A))$ by the proof above. Therefore $A$ is dual by Theorem 3.1.

4. A realization of the algebra $M_f(A)$.

**Theorem 4.1.** Let $A$ be a dual $A^*$-algebra of the first kind and let $\mathcal{A}$ be its
completion. Let $\pi_{\mathcal{A}}$ be the canonical map of $\mathcal{A}$ into $\mathcal{A}^{**}$. Then $\pi_{\mathcal{A}}(A)$ is an ideal
of $\mathcal{A}^{**}$ when $\mathcal{A}^{**}$ is given Arens product.

**Proof.** Let $x \in A, F \in \mathcal{A}^{**}$ and let $\{e_n\}$ be a maximal orthogonal family
of selfadjoint minimal idempotents in $A$. By [4, Theorem 16, p. 30], $\Sigma e_n x$ is
summable to $x$ in the norm $\| \cdot \|$, and hence there exists only a countable number of $e_n$
for which $e_n x \neq 0$, say $e_{a_1}, e_{a_2}, \ldots$. Since $A$ and $\mathcal{A}$ have the

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same socle and \( \pi_{\mathfrak{H}}(A) \) is an ideal of \( \mathfrak{H}^{**} \), it follows that \( F \ast \pi_{\mathfrak{H}}(e_\alpha) \in \pi_{\mathfrak{H}}(A) \) for \( i = 1, 2, \ldots \). Let \( m, n \) be positive integers, \( m < n \). By [4, Lemma 4, p. 18], we have

\[
\left\| \sum_{i=1}^{n} F \ast \pi_{\mathfrak{H}}(e_\alpha) \pi_{\mathfrak{H}}(x) - \sum_{i=1}^{m} F \ast \pi_{\mathfrak{H}}(e_\alpha) \ast \pi_{\mathfrak{H}}(x) \right\| \leq k \left\| \left( \sum_{i=m+1}^{n} F \ast \pi_{\mathfrak{H}}(e_\alpha) \right) \right\| \leq k |F| \left\| \sum_{i=m+1}^{n} \pi_{\mathfrak{H}}(e_\alpha) \right\| \leq k |F| \left\| \sum_{i=m+1}^{n} \pi_{\mathfrak{H}}(e_\alpha) \ast \pi_{\mathfrak{H}}(x) \right\|,
\]

where \( |F| \) denotes the norm of \( F \) in \( \mathfrak{H}^{**} \) and \( k \) is a positive constant. Thus \( \{ \sum_{i=1}^{n} F \ast \pi_{\mathfrak{H}}(e_\alpha) \ast \pi_{\mathfrak{H}}(x) \} \) is a Cauchy sequence in \( \pi_{\mathfrak{H}}(A) \) with respect to the norm \( \| \cdot \| \), and so there exists \( z \in A \) such that \( \pi_{\mathfrak{H}}(z) = \sum_{i=1}^{\infty} F \ast \pi_{\mathfrak{H}}(e_\alpha) \ast \pi_{\mathfrak{H}}(x) \). Since \( \sum_{i=1}^{\infty} F \ast \pi_{\mathfrak{H}}(e_\alpha) \ast \pi_{\mathfrak{H}}(x) \) also converges to \( \pi_{\mathfrak{H}}(z) \) and to \( F \ast \pi_{\mathfrak{H}}(x) \) in the norm \( \| \cdot \| \), we have \( \pi_{\mathfrak{H}}(z) = F \ast \pi_{\mathfrak{H}}(x) \). Hence \( F \ast \pi_{\mathfrak{H}}(x) \in \pi_{\mathfrak{H}}(A) \), for all \( x \in A \) and \( F \in \mathfrak{H}^{**} \). Similarly we can show that \( \pi_{\mathfrak{H}}(x) \ast F \in \mathfrak{H}^{**} \), for all \( x \in A \) and \( F \in \mathfrak{H}^{**} \). Therefore \( \pi_{\mathfrak{H}}(A) \) is an ideal of \( \mathfrak{H}^{**} \).

**Theorem 4.2.** Let \( A \) be a dual \( A^\ast \) algebra of the first kind and \( \mathfrak{H} \) its completion. Then \( M_{\mathfrak{H}}(A) \) is algebra isomorphic to \( \mathfrak{H}^{**} \) when \( \mathfrak{H}^{**} \) is given Arens product. This isomorphism is given by the following relation: For each \( T \in M_{\mathfrak{H}}(A) \) there exists a unique \( F_T \in \mathfrak{H}^{**} \) such that

\[
\pi_{\mathfrak{H}}(Tx) = F_T \ast \pi_{\mathfrak{H}}(x) \quad (x \in A).
\]

**Proof.** For each \( x \in A \), let \( \| x \|_A' = \sup\{\| xy \| : \| y \| < 1, y \in A \} \). Then \( \| \cdot \|_A' \) is a norm on \( A \) which is equivalent to \( \| \cdot \| \) [4, Theorem 18, p. 31]. Hence if \( T \in M_{\mathfrak{H}}(A) \) and \( x \in A \), then

\[
\| Tx \|_A' = \sup\{\| T(x)y \| : \| y \| < 1, y \in A \} = \sup\{\| T(xy) \| : \| y \| < 1, y \in A \} \leq k' \| T \| \| x \|,
\]

where \( k' \) is a constant \( > 0 \). Thus \( \| Tx \| \leq k'' \| x \| \) for all \( x \in A \) and some constant \( k'' > 0 \). Since \( A \) is dense in \( \mathfrak{H} \), it follows that \( T \) has a unique bounded extension \( T' \) to \( \mathfrak{H} \). Clearly \( T' \in M_{\mathfrak{H}}(\mathfrak{H}) \). By [2, Corollary 3.2, p. 509], there exists a unique \( F_T \in \mathfrak{H}^{**} \) such that \( \pi_{\mathfrak{H}}(Tx) = F_T \ast \pi_{\mathfrak{H}}(x) \) for all \( x \in A \).
Since, by Theorem 4.1, \( \pi_{\mathcal{H}}(A) \) is an ideal of \( \mathcal{H}^{**} \), we have that \( T \rightarrow F_T \) is an algebra isomorphism of \( M_{1}(A) \) onto \( \mathcal{H}^{**} \).

**References**