ON ODD DIMENSIONAL SURGERY WITH
FINITE FUNDAMENTAL GROUP

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Abstract. One proves that, for any finite group G and homomorphism \( \omega: G \to \mathbb{Z}/2\mathbb{Z} \), the natural homomorphism \( L_{2k+1}(\mathbb{Z}G, \omega) \to L_{2k+1}(\mathbb{Q}G, \omega) \) between Wall surgery groups is identically zero. Some results concerning the exponent of \( L_{2k+1}(\mathbb{Z}G; \omega) \) are deduced.

1. Introduction. Let \( G \) be a group and \( \omega: G \to \mathbb{Z}/2\mathbb{Z} \) be a homomorphism (orientation character). Let \( L_n(G; \omega) \) be the Wall surgery obstruction group in dimension \( n \) for surgery to a homotopy equivalence. By changing \( \mathbb{Z}G \) into \( \mathbb{Q}G \) in the definition of \( L_n \), one gets groups \( L_n(\mathbb{Q}G; \omega) \). They contain the obstruction for surgery to an \([n - 1]/2\)-connected rational homology equivalence. There is a natural homomorphism \( r: L_n(\mathbb{Z}G; \omega) \to L_n(\mathbb{Q}G; \omega) \) which is used in the long exact localization sequence for surgery obstruction groups of W. Pardon [P1].

Our first result is the following:

Theorem 1. For any finite group G and homomorphism \( \omega: G \to \mathbb{Z}/2\mathbb{Z} \), the homomorphism \( r: L_{2k+1}(\mathbb{Z}G; \omega) \to L_{2k+1}(\mathbb{Q}G; \omega) \) is the zero homomorphism.

Consequently, in the odd dimensional case with a finite fundamental group, the surgery to a rational homotopy equivalence is always possible. The obstruction for surgery to a homotopy equivalence is always expressible by the linking numbers approach developed in [KM], [W1] and [C]. Since the kernel of \( r \) is annihilated by 8 [C], one has

Corollary 2. For any finite group G and homomorphism \( \omega: G \to \mathbb{Z}/2\mathbb{Z} \), \( L_{2k+1}(\mathbb{Z}G; \omega) \) is annihilated by 8.

It has been conjectured that this exponent is 4, at least for a large class of finite groups. In this direction, a slight improvement of the proof of Theorem 1 gives the following results:

Theorem 3. Let \( 1 \to N \to G \to \Phi B \to 1 \) be an exact sequence of finite groups, and let \( \omega: B \to \mathbb{Z}/2\mathbb{Z} \) be a homomorphism. Suppose that \( N \) is a 2-group. If \( y \in \text{Ker} (\Phi_\ast: L_{2k+1}(G; \Phi \circ \omega) \to L_{2k+1}(B; \omega)) \), then \( 4y = 0 \).

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Corollary 4. For any finite 2-group $G$ and any homomorphism $\omega: G \to \mathbb{Z}/2\mathbb{Z}$, $L_{2k+1}^h(G; \omega)$ is annihilated by 4.

Corollary 5. Let $G$ be a finite group whose 2-Sylow subgroup is normal. Suppose that $\omega$ is trivial (the orientable case). Then $L_{2k+1}^h(G; \omega)$ is annihilated by 4.

For instance, the assumptions of Corollary 5 are fulfilled if $G$ is a finite nilpotent group, since a finite nilpotent group is the product of its Sylow subgroups [H].

A part of these statements is more or less known to be obtainable by different methods. Some particular cases are also deducible from published results of other authors. For instance, if $k$ is odd, Theorem 1 is obvious since $L_{2k+1}^h(\mathbb{Q}G; \omega) = 0$ [C]. On the other hand, in the orientable case ($\omega$ trivial), Corollary 2 can be deduced from [W3, exact sequence, p. 78 and remark (4), p. 2]. Corollaries 4 and 5 are deducible from [W3] when $G$ is abelian. Recently, W. Pardon independently found an algebraic proof of Corollary 4 and A. Bak computed $L_{2k}^h(G; \omega)$ when $G$ has its 2-Sylow subgroup normal and abelian [Bak2]. Finally, an announcement of Theorem 1 when $G$ is abelian was published by R. M. Geist [G]. Our proofs are independent from these results and our approach is quite different.

In §2, we give a sufficient condition for a degree one map between a manifold pair and a Poincaré pair (of odd dimension) to be a rational homotopy equivalence. In §3, we establish a functoriality property for a part of the localization exact sequence of surgery groups ([PI] and [P2]). It would be interesting to know in which generality such a functoriality property holds.

Results of §§2 and 3 are used in §4 for proving Theorem 1; finally, Theorem 4 and Corollaries 4 and 5 are proved in §5.

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2. A class of rational homology equivalences. Denote, as usual, by $\mathbb{Z}_{(p)}$ the subring of $\mathbb{Q}$ of fractions expressible with a denominator prime to $p$. This section is devoted to proving the following proposition:

Proposition 2.1. Let $(X, Y)$ be a Poincaré pair of dimension $2k + 1$ such that $\pi_1(Y) \cong \pi_1(X)$ is a finite group $G$. Let $N \subset G$ be a normal $p$-subgroup of $G$ ($p$ some prime). Consider a map $f: (M; \partial M) \to (X; Y)$ of degree one ($M^{2k+1}$ a compact manifold), with $f|\partial M$ a homotopy equivalence. Suppose that $f$ is $k$-connected and is a $\mathbb{Z}(G/N)$-homology equivalence (local coefficients). Then $f$ is a $\mathbb{Z}_{(p)}$-homology equivalence (and thus a rational homology equivalence).

Proof. If $B$ is a $\mathbb{Z}G$-module, we denote by $K_k(M; B)$ the $\mathbb{Z}G$-module $H_{k+1}(f; B)$ and $K_k(M)$ is used for $K_k(M; \mathbb{Z}G)$. By [CS, Lemma 1.4] one has $K_k(M; \mathbb{Z}(G/N)) = K_k(M) \otimes_{\mathbb{Z}G} \mathbb{Z}(G/N)$. This last module is $\mathbb{Z}$-isomorphic to $K_k(M) \otimes_{\mathbb{Z}N} \mathbb{Z} = K_k(\mathbb{Z})/I \cdot K_k(\mathbb{Z})$, where $I$ is the augmentation ideal of $N$. 

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Since $f$ is a $\mathbb{Z}(G/N)$-homology equivalence, one has $K_k(M; \mathbb{Z}(G/N)) = 0$, and then $K_k(M) = I \cdot K_k(M)$.

Since $K_k(M)$ is finitely generated [W2, Lemma 2.3] and $G$ is finite, $K_k(M)/pK_k(M)$ is a finite abelian $p$-group. The action of the finite $p$-group $N$ on any finite abelian $p$-group $L$ is nilpotent, i.e. $I^s \cdot L = 0$ for $s$ large enough. (The proof goes like in [H, pp. 47 and 155].) Therefore $K_k(M) = pK_k(M)$, which is equivalent to $K_k(M) \otimes_{\mathbb{Z}} \mathbb{Z}_p = 0$. But $K_k(M) \otimes_{\mathbb{Z}} \mathbb{Z}_p G = K_k(M; \mathbb{Z}_p G)$, the last isomorphism by [CS, Lemma 1.4]. Thus, $K_k(M; \mathbb{Z}_p G) = 0$ and $f$ is a $\mathbb{Z}_p$-homology equivalence.

3. Partial functoriality for the surgery group localization exact sequence. We will use a functorial property of the following leg of the Pardon exact sequence [P1]:

$$L_{2k+2}^h(QG, \omega) \rightarrow L_{2k+1}^l(ZG; \mathbb{Z} - \{0\})$$

$$\rightarrow L_{2k+1}^h(G, \omega) \rightarrow L_{2k+1}^h(QG, \omega).$$

This functoriality property is implied by the corresponding property for this other formulation of the sequence [P2, Theorem 2.1]:

$$W_0^{-\lambda}(QG) \rightarrow W_0^{-\lambda}(QG/ZG) \rightarrow W_1^{\lambda}(ZG) \rightarrow W_1^{\lambda}(QG)$$

where $\lambda = (-1)^k$. (See [P2] for the definitions.) Sequences (3.1) and (3.2) are related by the following commutative diagram:

$$W_0^{-\lambda}(QG) \rightarrow W_0^{-\lambda}(QG/ZG) \rightarrow W_1^{\lambda}(ZG) \rightarrow W_1^{\lambda}(QG)$$

$$L_{2k+2}^h(QG, \omega) \rightarrow L_{2k+1}^l(ZG; \mathbb{Z} - \{0\}) \rightarrow L_{2k+1}^h(G; \omega) \rightarrow L_{2k+1}^h(QG; \omega)$$

where the left two vertical arrows are identity maps (the groups are identical) and two right vertical arrows divide out by $w_i^\lambda = (\lambda_i^\lambda)$. Let $\mathcal{G}$ be the category whose objects are pairs $(G; \omega)$, where $G$ is a finite group and $\omega: G \rightarrow \mathbb{Z}/2\mathbb{Z}$ is a homomorphism. A morphism $F: (G, \omega_G) \rightarrow (H, \omega_H)$ of $\mathcal{G}$ is a homorphism $f: G \rightarrow H$ such that $\omega_G = \omega_H \circ f$. Let $\text{Ab}$ denote the category of abelian groups. The group rings $RG$ ($R = \mathbb{Z}$ or $Q$) are understood to be endowed with the involution $\sum n_g^g = \sum n_g^g \omega(g) g^{-1}$.

**Proposition 3.4.** The correspondences

$$(G, \omega) \rightarrow W_i^{\lambda}(RG), \quad i = 0 \text{ or } 1, \lambda = \pm 1,$$

and
(G, \omega) \mapsto W_0^\lambda(\mathbb{Q}G/\mathbb{Z}G)

give rise to functors from \mathcal{S} to \text{Ab} so that sequence (3.2) (and thus (3.1)) is functorial.

**Proof.** \( W_1^\lambda(\mathbb{R}G) \) are functors in the usual way. The only nonobvious point is to define \( \tilde{f}_z : W_0^\lambda(\mathbb{Q}G/\mathbb{Z}G) \to W_0^\lambda(\mathbb{Q}H/\mathbb{Z}H) \) for an \( \mathfrak{S} \)-morphism \( f : (G, \omega_G) \to (H, \omega_H) \). By definition of \( W_0^\lambda(\mathbb{Q}G/\mathbb{Z}G) \) [P2, p. 10] it suffices to have \( \tilde{f}_z \) defined on classes in \( W_0^\lambda \) which are represented by a triple \((M, \varphi, \psi)\), where

1. \( M \) is a finite \( \mathbb{Z}G \)-module admitting a short free \( \mathbb{Z}G \)-resolution

\[ 0 \to F_1 \xrightarrow{\mu} F_0 \to M \to 0. \]

2. \( \varphi : M \times M \to \mathbb{Q}G/\mathbb{Z}G \) is a nonsingular \( \lambda \)-hermitian form.

3. \( \psi : M \to \mathbb{Q}G/S_\lambda(\mathbb{Z}G) \) is a function satisfying (i)–(iii) of [P2].

Let us define the image by \( \tilde{f}_z \) of a class \((M, \varphi, \psi)\) to be the class of \((M', \varphi', \psi')\), where

1. \( M' = M \otimes_{\mathbb{Z}G} \mathbb{Z}H \).

2. \( \varphi' : M' \times M' \to \mathbb{Q}H/\mathbb{Z}H \) is defined by \( \varphi'(x \otimes a, y \otimes b) = \overline{\alpha}(\varphi(x, y)) b \)

(see [Ba, §6]).

3. \( \psi' : M' \to \mathbb{Q}H/S_\lambda(\mathbb{Z}H) \) is the unique extension of \( \psi \) such that \( \psi(x \otimes 1) = \overline{f}(\psi(x)) \) [Ba, 6.3].

Let us check that \((M', \varphi', \psi')\) actually determines a class in \( W_0^\lambda(\mathbb{Q}H/\mathbb{Z}H) \). \( M' \) is a finite \( \mathbb{Z}H \)-module; it admits the following short free resolution:

\[ 0 \to F_1 \otimes \mathbb{Z}H \xrightarrow{\mu \otimes 1} F_0 \otimes \mathbb{Z}H \to M' \to 0 \]

(the tensor product is always understood over \( \mathbb{Z}G \)). Indeed \( F_1 \otimes \mathbb{Z}H \) are \( \mathbb{Z}H \)-free of same \( \mathbb{Z}H \)-rank. Since \( H \) is a finite group, \( F_1 \otimes \mathbb{Z}H \) and \( F_2 \otimes \mathbb{Z}H \) have same finite \( \mathbb{Z} \)-rank. Thus \( \mu \otimes 1 \) is injective.

To prove that \( \varphi' \) is nonsingular, it suffices [Ba, p. 45] to check that the homomorphism

\[ j_M : \text{Hom}_{\mathbb{Z}G}(M; \mathbb{Q}G/\mathbb{Z}G) \otimes \mathbb{Z}H \to \text{Hom}_{\mathbb{Z}H}(M'; \mathbb{Q}H/\mathbb{Z}H) \]

given by \( j_M(h \otimes a)(x \otimes b) = \overline{\alpha}(h(x)) b \) is an isomorphism (observe that \( \mathbb{Q}G/\mathbb{Z}G \otimes \mathbb{Z}H \simeq \mathbb{Q}H/\mathbb{Z}H \)). By [P2, Proposition 1.4], \( \text{Hom}_{\mathbb{Z}G}(M; \mathbb{Q}G/\mathbb{Z}G) \) has the following free resolution:

\[ 0 \to \text{Hom}_{\mathbb{Z}G}(F_0; \mathbb{Z}G) \xrightarrow{\beta} \text{Hom}_{\mathbb{Z}G}(F_1; \mathbb{Z}G) \to \text{Hom}_{\mathbb{Z}G}(M; \mathbb{Q}G/\mathbb{Z}G) \to 0. \]

As above, tensoring by \( \mathbb{Z}H \) leaves this sequence exact. Hence, one has

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The commutativity of the lowest square comes from the definitions of $j_{F_0}$ and $j_{M}$ and of the identification of $\text{coker } \mu$ with $\text{Hom}_{ZG}(M; QG/ZG)$ [P2, (1.5)]. Clearly, $j_{ZG}$ is an isomorphism. By additivity, $j_{F_0}$ and $j_{F_1}$ are isomorphisms. Thus, $j_{M}$ is an isomorphism.

Finally, one checks without difficulty that a kernel is mapped onto a kernel. Thus $f_\lambda : \text{W}_0^\lambda(QG/ZG) \to \text{W}_0^\lambda(QH/ZH)$ is well defined.

The functoriality of sequence (3.2) then comes directly from the definitions of each arrow.

4. Proof of Theorem 1. Theorem 1 will be proven first when $G$ is a finite 2-group, then when $G$ is a 2-hyperelementary group and last in the general case using the induction theorem due to Dress. We shall make use of the following lemma, in which $r : L_{2k+1}(G; \Phi \circ \omega) \to L_{2k+1}(QG; \Phi \circ \omega)$ is the natural homomorphism, as in the Introduction.

**Lemma 4.1.** Let $1 \to N \to G \to B \to 1$ be an extension of finite groups, with $N$ a $p$-group. Let $\omega : B \to Z/2Z$ be a homomorphism. Then, for all $x \in \text{Ker} \ (L_{2k+1}^h(G; \omega \circ \Phi) \to L_{2k+1}^h(B; \omega))$, one has $r(x) = 0$.

**Proof.** Let $M^{2k}$ be a manifold, $k \geq 3$, with $\pi_1(M) = G$ and orientation homomorphism $\omega \circ \Phi$. Accordingly [W2, Theorem 6.5] there exists a cobordism $(W^{2k+1}, M, M')$ and a normal map of degree one $(f, b) : (W, M, M') \to (M \times I, M \times \{0\}, M \times \{1\})$ such that $f|M = \text{id}_M$, the map $f|M'$ is a homotopy equivalence, and the surgery obstruction $\theta(f, b) \in L_{2k+1}(G; \omega \circ \Phi)$ is equal to $x$.

Let $\Gamma_{2k+1}^h(ZG \to \Phi ZB)$ be the Cappell-Shaneson surgery obstruction group [CS]. This is a subgroup of $L_{2k+1}^h(B; \omega)$. Therefore, the assumption on $x$ implies that $x$ belongs to the kernel of the natural homomorphism

$$L_{2k+1}^h(G; \omega \circ \Phi) \to \Gamma_{2k+1}^h(ZG \to \Phi ZB).$$
Hence, by [CS, Proposition 2.1], \((f, b)\) is normally cobordant to \((f', b')\) where \(f'\) is a \(k\)-connected \(\mathbb{Z}B\)-homology equivalence. By Proposition 2.1, \(f'\) is a rational homology equivalence and then, by [C, Theorem 4.10], \(r(x) = 0\).

**Proof of Theorem 1.**

*Case 1.* \(G\) is a finite 2-group. If \(\omega\) is the trivial homomorphism, the result follows from Lemma 4.1 applied to \(N = G\). If \(\omega\) is not trivial, one uses Lemma 4.1 applied to the case \(B = \mathbb{Z}/2\mathbb{Z}\) and \(\Phi = \omega_G\) together with the fact that \(L_{2k+1}^h(\mathbb{Z}/2\mathbb{Z}; \text{id}) = 0\) [W2, Theorem 13.A.1].

*Case 2.* \(G\) is a 2-hyperelementary group. This means that \(G\) has a decomposition \(1 \to C \to G \to P \to 1\), where \(P\) is a 2-group and \(C\) is cyclic. The proof will be by induction on \(d(G)\), the greatest odd divisor of \(|G|\). If \(d(G) = 1\), \(G\) is a 2-group and we can use Case 1. If \(d > 1\), one may write \(C = C_1 \times C_2\), where \(C_1\) is a cyclic \(p\)-group with \(p \neq 2\), and \((p, |C_2|) = 1\). Thus, one has a split exact sequence:

\[
1 \longrightarrow C_1 \longrightarrow G \xrightarrow{\Phi} \overline{G} \longrightarrow 1
\]

and \(\omega_G: G \to \mathbb{Z}/2\mathbb{Z}\) factors through \(\omega_{\overline{G}}: \overline{G} \to \mathbb{Z}/2\mathbb{Z}\). \(\overline{G}\) is a 2-hyperelementary group with \(d(\overline{G}) < d(G)\). By induction hypothesis, \(r: L_{2k+1}^h(G; \omega_G) \to L_{2k+1}^h(Q\overline{G}; \omega_{\overline{G}})\) is zero. Using the functoriality of sequence (3.1), one gets a diagram:

\[
\begin{array}{ccc}
L_{2k+1}^h(ZG; Z - \{0\}) & \overset{i}{\longrightarrow} & L_{2k+1}^h(G; \omega_G) \overset{r}{\longrightarrow} L_{2k+1}^h(QG; \omega_G) \\
\Phi \downarrow & & \Phi \downarrow \\
L_{2k+1}^h(\overline{G}; Z - \{0\}) & \overset{i}{\longrightarrow} & L_{2k+1}^h(\overline{G}; \omega_{\overline{G}}) \overset{r}{\longrightarrow} L_{2k+1}^h(Q\overline{G}; \omega_{\overline{G}}) \\
\end{array}
\]

Thus one can write \(x \in L_{2k+1}^h(G; \omega_G)\) as \(x = x_1 + x_2\) with \(x_1 = i(y)\) and \(x_2 \in \ker \Phi_*\). By Lemma 4.1, \(r(x) = 0\).

*Case 3.* General case. Since \(L_{2k+1}^h(G; \omega_G)\) is a 2-group [C], the Dress induction theorem [D] asserts that the product of restrictions \(L_{2k+1}^h(G; \omega) \to \prod_{H \in \mathfrak{X}(G)} L_{2k+1}^h(H; \omega|H)\) is injective, where \(\mathfrak{X}(G)\) is the set of 2-hyperelementary subgroups of \(G\). The same holds for \(L_{2k+1}^h(QG, \omega)\). Theorem 1 then follows, using Case 2.

5. **Proof of Theorem 3 and Corollaries 4 and 5.** Let \(M^{2k}\) be a closed manifold with \(\pi_1(M) = G\) and orientation character \(\omega \circ \Phi\). Represent \(y\) as the surgery obstruction for a normal map of degree one:

\[
f: (W^{2k+1}, M, M') \to (M \times I, M \times \{0\}, M \times \{1\}).
\]

As in the proof of Lemma 4.1, the condition on \(y\) implies that \(f\) is normally cobordant to an \(f'\) that is a \(k\)-connected \(\mathbb{Z}B\)-homology equivalence. By Proposition 2.1, \(f'\) is a \(\mathbb{Z}_{(2)}\)-homology equivalence. Therefore, using [PP, §5], one can consider \(y\) as an element of \(J_k(G)\), the Grothendieck group based on...
linking forms over finite $\mathbb{Z}G$-modules without 2-torsion. $J_k(G)$ has exponent 4 [PP, Theorem 5.1]. Thus $4v = 0$.

Corollary 4 is a consequence of Theorem 4, applied to $N = G$ in the orientable case and to $\Phi = \omega$ otherwise (see the proof of Theorem 1, Case 1). For Corollary 5, one uses Theorem 3 for $N$ equal to the normal 2-Sylow group of $G$, and the fact that $L^k_{\text{odd}}(H) = 0$ when $H$ has odd order ([Bak] or [P]).

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