ON ODD DIMENSIONAL SURGERY WITH
FINITE FUNDAMENTAL GROUP

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Abstract. One proves that, for any finite group $G$ and homomorphism
$\omega: G \to \mathbb{Z}/2\mathbb{Z}$, the natural homomorphism $L_{2k+1}^h(\mathbb{Z}G, \omega) \to L_{2k+1}^h(\mathbb{Q}G, \omega)$
between Wall surgery groups is identically zero. Some results concerning the
exponent of $L_{2k+1}^h(\mathbb{Z}G; \omega)$ are deduced.

1. Introduction. Let $G$ be a group and $\omega: G \to \mathbb{Z}/2\mathbb{Z}$ be a homomorphism
(orientation character). Let $L_n^h(G; \omega)$ be the Wall surgery obstruction group in
dimension $n$ for surgery to a homotopy equivalence. By changing $\mathbb{Z}G$ into $\mathbb{Q}G$
in the definition of $L_n^h$, one gets groups $L_n^h(\mathbb{Q}G; \omega)$. They contain the
obstruction for surgery to an $[n-1]/2$-connected rational homology equiva-
lence. There is a natural homomorphism $r: L_n^h(G; \omega) \to L_n^h(\mathbb{Q}G; \omega)$ which is
used in the long exact localization sequence for surgery obstruction groups of
W. Pardon [P1].

Our first result is the following:

**Theorem 1.** For any finite group $G$ and homomorphism $\omega: G \to \mathbb{Z}/2\mathbb{Z}$, the
homomorphism $r: L_{2k+1}^h(G; \omega) \to L_{2k+1}^h(\mathbb{Q}G; \omega)$ is the zero homomorphism.

Consequently, in the odd dimensional case with a finite fundamental group,
the surgery to a rational homotopy equivalence is always possible. The
obstruction for surgery to a homotopy equivalence is always expressible by the
linking numbers approach developed in [KM], [W1] and [C]. Since the kernel
of $r$ is annihilated by 8 [C], one has

**Corollary 2.** For any finite group $G$ and homomorphism $\omega: G \to \mathbb{Z}/2\mathbb{Z}$,
$L_{2k+1}^h(G; \omega)$ is annihilated by 8.

It has been conjectured that this exponent is 4, at least for a large class of
finite groups. In this direction, a slight improvement of the proof of Theorem
1 gives the following results:

**Theorem 3.** Let $1 \to N \to G \to B \to 1$ be an exact sequence of finite groups,
and let $\omega: B \to \mathbb{Z}/2\mathbb{Z}$ be a homomorphism. Suppose that $N$ is a 2-group. If
$y \in \text{Ker}(\Phi_\ast: L_{2k+1}^h(G; \Phi \circ \omega) \to L_{2k+1}^h(B; \omega))$, then $4y = 0$.  

Received by the editors May 13, 1976 and, in revised form, July 30, 1976.
1 Supported in part by NSF Grant MPS72-05055 A03.
Corollary 4. For any finite 2-group $G$ and any homomorphism $\omega: G \rightarrow \mathbb{Z}/2\mathbb{Z}$, $L_{2k+1}^h(G; \omega)$ is annihilated by 4.

Corollary 5. Let $G$ be a finite group whose 2-Sylow subgroup is normal. Suppose that $\omega$ is trivial (the orientable case). Then $L_{2k+1}^h(G; \omega)$ is annihilated by 4.

For instance, the assumptions of Corollary 5 are fulfilled if $G$ is a finite nilpotent group, since a finite nilpotent group is the product of its Sylow subgroups [H].

A part of these statements is more or less known to be obtainable by different methods. Some particular cases are also deducible from published results of other authors. For instance, if $k$ is odd, Theorem 1 is obvious since $L_{2k+1}^h(\mathbb{Q}G; \omega) = 0$ [C]. On the other hand, in the orientable case ($\omega$ trivial), Corollary 2 can be deduced from [W3, exact sequence, p. 78 and remark (4), p. 2]. Corollaries 4 and 5 are deducible from [W3] when $G$ is abelian. Recently, W. Pardon independently found an algebraic proof of Corollary 4 and A. Bak computed $L_{2k}^h(G; \omega)$ when $G$ has its 2-Sylow subgroup normal and abelian [Bak2]. Finally, an announcement of Theorem 1 when $G$ is abelian was published by R. M. Geist [G]. Our proofs are independent from these results and our approach is quite different.

In §2, we give a sufficient condition for a degree one map between a manifold pair and a Poincaré pair (of odd dimension) to be a rational homotopy equivalence. In §3, we establish a functoriality property for a part of the localization exact sequence of surgery groups ([P1] and [P2]). It would be interesting to know in which generality such a functoriality property holds.

Results of §§2 and 3 are used in §4 for proving Theorem 1; finally, Theorem 4 and Corollaries 4 and 5 are proved in §5.

I am grateful to W. Pardon for conversations, to C.T.C. Wall and A. Bak for commenting on these results, and to the referee for a great simplification of my original proof of Proposition 2.1.

2. A class of rational homology equivalences. Denote, as usual, by $\mathbb{Z}_{(p)}$ the subring of $\mathbb{Q}$ of fractions expressible with a denominator prime to $p$. This section is devoted to proving the following proposition:

Proposition 2.1. Let $(X, Y)$ be a Poincaré pair of dimension $2k + 1$ such that $\pi_1(Y) \simeq \pi_1(X)$ is a finite group $G$. Let $N \subseteq G$ be a normal $p$-subgroup of $G$ ($p$ some prime). Consider a map $f: (M; \partial M) \rightarrow (X; Y)$ of degree one ($M^{2k+1}$ a compact manifold), with $f|\partial M$ a homotopy equivalence. Suppose that $f$ is $k$-connected and is a $\mathbb{Z}(G/N)$-homology equivalence (local coefficients). Then $f$ is a $\mathbb{Z}_{(p)}$-homology equivalence (and thus a rational homology equivalence).

Proof. If $B$ is a $\mathbb{Z}G$-module, we denote by $K_k(M; B)$ the $\mathbb{Z}G$-module $H_{k+1}(f; B)$ and $K_k(M)$ is used for $K_k(M; \mathbb{Z}G)$. By [CS, Lemma 1.4] one has $K_k(M; \mathbb{Z}(G/N)) = K_k(M) \otimes_{\mathbb{Z}G} \mathbb{Z}(G/N)$. This last module is $\mathbb{Z}$-isomorphic to $K_k(M) \otimes_{\mathbb{Z}N} \mathbb{Z} = K_k(M)/I \cdot K_k(M)$, where $I$ is the augmentation ideal of $N$. 

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Since $f$ is a $\mathbb{Z}(G/N)$-homology equivalence, one has $K_k(M; \mathbb{Z}(G/N)) = 0$, and then $K_k(M) = I \cdot K_k(M)$.

Since $K_k(M)$ is finitely generated [W2, Lemma 2.3] and $G$ is finite, $K_k(M)/pK_k(M)$ is a finite abelian $p$-group. The action of the finite $p$-group $N$ on any finite abelian $p$-group $L$ is nilpotent, i.e. $I^s \cdot L = 0$ for $s$ large enough. (The proof goes like in [H, pp. 47 and 155].) Therefore $K_k(M) = pK_k(M)$, which is equivalent to $K_k(M) \otimes_{\mathbb{Z}} \mathbb{Z}_p G = 0$. But $K_k(M) \otimes_{\mathbb{Z}} \mathbb{Z}_p G = K_k(M; \mathbb{Z}_p G)$, the last isomorphism by [CS, Lemma 1.4]. Thus, $K_k(M; \mathbb{Z}_p G) = 0$ and $f$ is a $\mathbb{Z}(p)$-homology equivalence.

3. Partial functoriality for the surgery group localization exact sequence. We will use a functorial property of the following leg of the Pardon exact sequence [P1]:

$$L_{2k+2}^h(QG, \omega) \rightarrow L_{2k+1}^i(ZG; \mathbb{Z} - \{0\})$$

(3.1)

$$\rightarrow L_{2k+1}^h(G, \omega) \rightarrow L_{2k+1}^h(QG, \omega).$$

This functoriality property is implied by the corresponding property for this other formulation of the sequence [P2, Theorem 2.1]:

$$W_0^{-\lambda}(QG) \rightarrow W_0^{-\lambda}(QG/ZG) \rightarrow W_1^{\lambda}(ZG) \rightarrow W_1^{\lambda}(QG)$$

(3.2)

where $\lambda = (-1)^k$. (See [P2] for the definitions.) Sequences (3.1) and (3.2) are related by the following commutative diagram:

(3.3)

where the left two vertical arrows are identity maps (the groups are identical) and two right vertical arrows divide out by $w_1^\lambda = \left(\frac{1}{\lambda}^{1/2}\right)$.

Let $\mathcal{F}$ be the category whose objects are pairs $(G; \omega)$, where $G$ is a finite group and $\omega: G \rightarrow \mathbb{Z}/2\mathbb{Z}$ is a homomorphism. A morphism $F: (G, \omega_G) \rightarrow (H, \omega_H)$ of $\mathcal{F}$ is a homorphism $f: G \rightarrow H$ such that $\omega_G = \omega_H \circ f$. Let $Ab$ denote the category of abelian groups. The group rings $RG$ ($R = \mathbb{Z}$ or $Q$) are understood to be endowed with the involution $\sum n_g \bar{g} = \sum n_g \omega(g)g^{-1}$.

**Proposition 3.4.** The correspondences

$$(G, \omega) \rightarrow W_i^\lambda(RG), \quad i = 0 \text{ or } 1, \lambda = \pm 1,$$

and
give rise to functors from $\mathcal{F}$ to $\text{Ab}$ so that sequence \( (3.2) \) (and thus \( (3.1) \)) is functorial.

**Proof.** $W_1^\lambda (RG)$ are functors in the usual way. The only nonobvious point is to define $f_*: W_0^\lambda (QG/ZG) \to W_0^\lambda (QH/ZH)$ for an $\mathcal{F}$-morphism $f: (G, \omega_G) \to (H, \omega_H)$. By definition of $W_0^\lambda (QG/ZG)$ \cite[p. 10]{P2} it suffices to have $f_*$ defined on classes in $W_0^\lambda$ which are represented by a triple $(M, \varphi, \psi)$, where

1. $M$ is a finite $ZG$-module admitting a short free $ZG$-resolution

$$0 \to F_1 \to F_0 \to M \to 0.$$ 

(2) $\varphi: M \times M \to QG/ZG$ is a nonsingular $\lambda$-hermitian form.

(3) $\psi: M \to QG/S_\lambda (ZG)$ is a function satisfying (i)–(iii) of \cite{P2}.

Let us define the image by $f_*$ of a class $(M, \varphi, \psi)$ to be the class of $(M', \varphi', \psi')$, where

1. $M' = M \otimes_{ZG} ZH$.

2. $\varphi': M' \times M' \to QH/ZH$ is defined by $\varphi'(x \otimes a, y \otimes b) = \bar{\alpha} f(\varphi(x, y)) b$ (see \cite[§6]{Ba}).

3. $\psi': M' \to QH/S_\lambda (ZH)$ is the unique extension of $\psi$ such that $\psi'(x \otimes 1) = f(\psi(x))$ \cite[6.3]{Ba}.

Let us check that $(M', \varphi', \psi')$ actually determines a class in $W_0^\lambda (QH/ZH)$. $M'$ is a finite $ZH$-module; it admits the following short free resolution:

$$0 \to F_1 \otimes ZH \xrightarrow{\mu \otimes 1} F_0 \otimes ZH \to M' \to 0$$

(the tensor product is always understood over $ZG$). Indeed $F_1 \otimes ZH$ are $ZH$-free of same $ZH$-rank. Since $H$ is a finite group, $F_1 \otimes ZH$ and $F_2 \otimes ZH$ have same finite $Z$-rank. Thus $\mu \otimes 1$ is injective.

To prove that $\varphi'$ is nonsingular, it suffices \cite[p. 45]{Ba} to check that the homomorphism

$$j_{M'}: \text{Hom}_{ZG} (M; QG/ZG) \otimes ZH \to \text{Hom}_{ZH} (M'; QH/ZH)$$

given by $j_M(h \otimes a)(x \otimes b) = \bar{\alpha} f(h(x)) b$ is an isomorphism (observe that $QG/ZG \otimes ZH \cong QH/ZH$). By \cite[Proposition 1.4]{P2}, $\text{Hom}_{ZG} (M; QG/ZG)$ has the following free resolution:

$$0 \to \text{Hom}_{ZG} (F_0; ZG) \xrightarrow{\bar{\alpha}} \text{Hom}_{ZG} (F_1; ZG) \to \text{Hom}_{ZG} (M; QG/ZG) \to 0.$$ 

As above, tensoring by $ZH$ leaves this sequence exact. Hence, one has
The commutativity of the lowest square comes from the definitions of \( j_F \) and \( j_M \) and of the identification of \( \text{coker } \mu \) with \( \text{Hom}_{ZG}(M; QG/ZG) \) \([P2, (1.5)]\). Clearly, \( j_{ZG} \) is an isomorphism. By additivity, \( j_{F_0} \) and \( j_{F_1} \) are isomorphisms. Thus, \( j_M \) is an isomorphism.

Finally, one checks without difficulty that a kernel is mapped onto a kernel. Thus \( f_\lambda : W_0^h(QG/ZG) \rightarrow W_0^h(QH/ZH) \) is well defined.

The functoriality of sequence (3.2) then comes directly from the definitions of each arrow.

4. Proof of Theorem 1. Theorem 1 will be proven first when \( G \) is a finite 2-group, then when \( G \) is a 2-hyperelementary group and last in the general case using the induction theorem due to Dress. We shall make use of the following lemma, in which \( r: L_{2k+1}^h(G; \Phi \circ \omega) \rightarrow L_{2k+1}^h(QG; \Phi \circ \omega) \) is the natural homomorphism, as in the Introduction.

**Lemma 4.1.** Let \( 1 \rightarrow N \rightarrow G \rightarrow B \rightarrow 1 \) be an extension of finite groups, with \( N \) a \( p \)-group. Let \( \omega: B \rightarrow Z/2Z \) be a homomorphism. Then, for all \( x \in \text{Ker } (L_{2k+1}^h(G; \omega \circ \Phi) \rightarrow L_{2k+1}^h(B; \omega)) \), one has \( r(x) = 0 \).

**Proof.** Let \( M^{2k} \) be a manifold, \( k \geq 3 \), with \( \pi_1(M) = G \) and orientation homomorphism \( \omega \circ \Phi \). Accordingly \([W2, \text{Theorem 6.5}]\) there exists a cobordism \((W^{2k+1}, M, M')\) and a normal map of degree one \((f, b): (W, M, M') \rightarrow (M \times I, M \times \{0\}, M \times \{1\})\) such that \( f|M = \text{id}_M \), the map \( f|M' \) is a homotopy equivalence, and the surgery obstruction \( \theta(f, b) \in L_{2k+1}^h(G; \omega \circ \Phi) \) is equal to \( x \).

Let \( \Gamma_{2k+1}^h(ZG \rightarrow \Phi ZB) \) be the Cappell-Shaneson surgery obstruction group \([CS]\). This is a subgroup of \( L_{2k+1}^h(B; \omega) \). Therefore, the assumption on \( x \) implies that \( x \) belongs to the kernel of the natural homomorphism

\[
L_{2k+1}^h(G; \omega \circ \Phi) \rightarrow \Gamma_{2k+1}^h(ZG \rightarrow \Phi ZB).
\]
Hence, by [CS, Proposition 2.1], \((f', b')\) is normally cobordant to \((f', b')\) where \(f'\) is a \(k\)-connected \(\mathbb{Z}B\)-homology equivalence. By Proposition 2.1, \(f'\) is a rational homology equivalence and then, by [C, Theorem 4.10], \(r(x) = 0\).

**Proof of Theorem 1.**

Case 1. \(G\) is a finite 2-group. If \(\omega\) is the trivial homomorphism, the result follows from Lemma 4.1 applied to \(N = G\). If \(\omega\) is not trivial, one uses Lemma 4.1 applied to the case \(B = \mathbb{Z}/2\mathbb{Z}\) and \(\Phi = \omega_G\) together with the fact that \(L^h_{2k+1}(\mathbb{Z}/2\mathbb{Z}; \text{id}) = 0\) [W2, Theorem 13.A.1].

Case 2. \(G\) is a 2-hyperelementary group. This means that \(G\) has a decomposition \(1 \to C \to G \to P \to 1\), where \(P\) is a 2-group and \(C\) is cyclic. The proof will be by induction on \(d(G)\), the greatest odd divisor of \(|G|\). If \(d(G) = 1\), \(G\) is a 2-group and we can use Case 1. If \(d > 1\), one may write \(C = C_1 \times C_2\), where \(C_1\) is a cyclic \(p\)-group with \(p \neq 2\), and \((p, |C_2|) = 1\). Thus, one has a split exact sequence:

\[
1 \longrightarrow C_1 \longrightarrow G \xrightarrow{\Phi} \overline{G} \longrightarrow 1
\]

and \(\omega_G: G \to \mathbb{Z}/2\mathbb{Z}\) factors through \(\omega_{\overline{G}}: \overline{G} \to \mathbb{Z}/2\mathbb{Z}\). \(\overline{G}\) is a 2-hyperelementary group with \(d(\overline{G}) < d(G)\). By induction hypothesis, \(r: L^h_{2k+1}(\overline{G}; \omega_{\overline{G}}) \to L^h_{2k+1}(Q\overline{G}; \omega_{\overline{G}})\) is zero. Using the functoriality of sequence (3.1), one gets a diagram:

\[
\begin{array}{ccc}
L^f_{2k+1}(ZG; Z - \{0\}) & \overset{i}{\longrightarrow} & L^h_{2k+1}(G; \omega_G) \\
\Phi^f & \downarrow s^f & \Phi_* \\
L^s_{2k+1}(Z\overline{G}; Z - \{0\}) & \overset{i}{\longrightarrow} & L^h_{2k+1}(\overline{G}, \omega_{\overline{G}}) \\
\end{array}
\]

Thus one can write \(x \in L^h_{2k+1}(G; \omega_G)\) as \(x = x_1 + x_2\) with \(x_1 = i(y)\) and \(x_2 \in \ker \Phi_*\). By Lemma 4.1, \(r(x) = 0\).

Case 3. General case. Since \(L^h_{\text{odd}}(G; \omega)\) is a 2-group [C], the Dress induction theorem [D] asserts that the product of restrictions \(L^h_{2k+1}(G; \omega) \to \prod_{H \in \mathcal{X}(G)} L^h_{2k+1}(H; \omega|H)\) is injective, where \(\mathcal{X}(G)\) is the set of 2-hyperelementary subgroups of \(G\). The same holds for \(L^h_{2k+1}(QG, \omega)\). Theorem 1 then follows, using Case 2.

5. Proof of Theorem 3 and Corollaries 4 and 5. Let \(M^{2k}\) be a closed manifold with \(\pi_1(M) = G\) and orientation character \(\omega \circ \Phi\). Represent \(y\) as the surgery obstruction for a normal map of degree one:

\[
f: (W^{2k+1}, M, M') \to (M \times I, M \times \{0\}, M \times \{1\}).
\]

As in the proof of Lemma 4.1, the condition on \(y\) implies that \(f\) is normally cobordant to an \(f'\) that is a \(k\)-connected \(\mathbb{Z}B\)-homology equivalence. By Proposition 2.1, \(f'\) is a \(\mathbb{Z}_{(2)}\)-homology equivalence. Therefore, using \([PP, \S5]\), one can consider \(y\) as an element of \(J_k(G)\), the Grothendieck group based on
linking forms over finite $\mathbb{Z}G$-modules without 2-torsion. $J_k(G)$ has exponent 4 [PP, Theorem 5.1]. Thus $4v = 0$.

Corollary 4 is a consequence of Theorem 4, applied to $N = G$ in the orientable case and to $\Phi = \omega$ otherwise (see the proof of Theorem 1, Case 1). For Corollary 5, one uses Theorem 3 for $N$ equal to the normal 2-Sylow group of $G$, and the fact that $L^k_{\text{odd}}(H) = 0$ when $H$ has odd order ([Bak] or [P1]).

REFERENCES


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