DENSE SUBGROUPS OF LIE GROUPS

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Abstract. Let $G$ be a dense analytic subgroup with compact center of an analytic group $L$. Then there exist closed vector subgroups $W$ and $U$ of $G$ and a $(CA)$ closed normal analytic subgroup $M$ of $G$, which contains the center of $G$, such that $G = MWU$, $MW \cap U = M \cap W = \{e\}$, and $WU$ is a closed vector subgroup of $G$. Moreover, $L = MWU$, where $MW$ is a closed normal analytic subgroup of $L$ and $U$ is a toral group, such that $MW \cap U$ is finite.

1. Introduction. By an analytic group and an analytic subgroup of a Lie group, we mean a connected Lie group and a connected Lie subgroup, respectively. If $G$ and $H$ are Lie groups and $\varphi$ is a one-to-one (continuous) homomorphism from $G$ into $H$, $\varphi$ will be called an immersion. $\varphi$ will be called closed or dense, as $\varphi(G)$ is closed or dense in $H$. $G_0$ and $Z(G)$ will denote the identity component group and center of $G$, respectively.

If $G$ is an analytic group, $A(G)$ will denote the Lie group of all (bicontinuous) automorphisms of $G$, topologized with the generalized compact-open topology. $G$ will be called $(CA)$ if $I(G)$, the Lie group of all inner automorphisms of $G$, is closed in $A(G)$. It is well known that $G$ is $(CA)$ if and only if its universal covering group is $(CA)$.

If $G$ is a normal analytic subgroup of an analytic group $H$, then each element $h$ of $H$ induces an automorphism of $G$, namely, $g \mapsto hgh^{-1}$. We will denote this homomorphism from $H$ into $A(G)$ by $\rho_{GH}$. $I_H(h)$ will denote the inner automorphism of $H$ determined by $h \in H$. More generally, if $A$ is a subset of $H$, $I_H(A)$ will denote the set of all inner automorphisms of $H$ determined by elements of $A$. $I_H(H)$ will be written as $I(H)$, and the mapping $h \mapsto I_H(h)$ of $H$ onto $I(H)$ will be denoted by $I_H$.

If $N$ is an analytic group and $H$ is an analytic subgroup of $A(N)$, then $N \oplus H$ will denote the semidirect product of $N$ and $H$. On the other hand, if $G$ is an analytic group containing a closed normal analytic subgroup $N$ and a closed analytic subgroup $H$, such that $G = NH$, $N \cap H = \{e\}$, and such that the restriction of $\rho_{NG}$ to $H$ is one-to-one, we will frequently identify $G$ with $N \oplus \rho_{NG}(H)$ and $H$ with $\rho_{NG}(H)$, that is, we may write $G = N \oplus H$.

In Zerling [3] we proved the following theorem.

Main structure theorem. Let $G$ be a non-(CA) analytic group. Then there exist a (CA) analytic group $M$, a toral group $T$ in $A(M)$, and a dense vector...
subgroup $V$ of $T$, such that:

(i) $H = M \circledast T$ is a (CA) analytic group.

(ii) $G$ is isomorphic to the dense analytic subgroup $M \circledast V$ of $H$.

(iii) $\text{Z}(G)$ is contained in $M$.

(iv) $\text{Z}_0(G) = \text{Z}_0(H)$, and $\pi(\text{Z}(H))$ is finite, where $\pi$ is the natural projection of $H$ onto $T$. Moreover, if $G/\text{Z}(G)$ is homeomorphic to Euclidean space, then $\text{Z}(G) = \text{Z}(H)$.

(v) Each automorphism $\sigma$ of $G$ can be extended to an automorphism $\epsilon(\sigma)$ of $H$, such that $\epsilon: A(G) \to A(H)$ is a closed immersion.

We will now use this theorem in §2 to obtain our main results.

2. Main results.

**Lemma.** Let us maintain the notation of the main structure theorem and let $f: G \to L$ be a dense immersion of $G$ into an analytic group $L$. If $\text{Z}(G)$ is compact, then $f(M)$ is closed in $L$.

**Proof.** Since $G$ is non-(CA) we can appeal to Goto [1]: Let $N$ be a maximal analytic subgroup of $I(G)$, which contains the commutator subgroup of $I(G)$ and is closed in $A(G)$. Then there is a closed vector subgroup $V'$ of $I(G)$, such that $I(G) = NV'$, $N \cap V' = \{e\}$, and $\overline{I(G)} = N \cdot V'$, where $T' = \overline{V'}$ is a toral group. Moreover, $N \cap T'$ is finite, and the space of $\overline{I(G)}$ is diffeomorphic to the product space $N \times T'$.

In the proof of the main structure theorem in Zerling [3], $H$ is constructed in such a way that $\rho_{GH}(M) = N$, $\rho_{GH}(V) = V'$, and $\rho_{GH}(T) = T'$. Also, $\rho_{GH}$ is 1-1 on $T$.

Since $\text{Z}(G)$ is of finite index in $\text{Z}(H)$ from Zerling [4, Lemma 2.1], $\text{Z}(H)$ is also compact. Consider the normal analytic subgroup $\overline{f(M)} \cdot f(V)$ of $L$. Since the inner automorphic action of $f(V)$ on $\overline{f(M)}$ is effective, we have the Lie group $P = \overline{f(M)} \circledast V$. The image of each one-parameter subgroup of $V$ under $\rho_{GP}$ is not closed in $A(G)$. Therefore, since $G$ is dense in $P$, we see from Lemma 3.1 of Zerling [4] that the closure of $V$ in $A(\overline{f(M)})$ is a toral group, which we will denote by $T_1$.

Now let $Q = \overline{f(M)} \circledast T_1$. Then $\rho_{GQ}(T_1) = T'$, and since $\tau_1 \cdot (m, v) \cdot \tau_1^{-1} = (\tau_1(m), v)$ for all $(m, v)$ in $G$, we see that $\rho_{GQ}$ is 1-1 on $T_1$. Since $\rho_{GH}(T) = T'$, and $\rho_{GH}$ is 1-1 on $T$, we have that $\rho_{GQ}^{-1} \circ \rho_{GH}$ is an isomorphism of $T$ onto $T_1$. Hence, $H = M \circledast T$ is a dense (CA) analytic subgroup of $Q$. Since $\text{Z}(H)$ is compact, we may appeal to van Est [2, Theorem 2.2.1] to conclude that $H \cong Q$. Hence, $f(M) = \overline{f(M)}$ and our theorem is proved.

**Theorem.** Let $f: G \to L$ be a proper dense immersion of an analytic group $G$ into an analytic group $L$. Suppose $\text{Z}(G)$ is compact. Then there exist closed vector subgroups $W$ and $U$ of $G$ and a (CA) closed normal analytic subgroup $M$ of $G$, which contains $\text{Z}(G)$, such that:

(i) $G = MWU$, $MW \cap U = M \cap W = \{e\}$, and $WU$ is a closed vector subgroup of $G$. 

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(ii) \( L = f(MW) \cdot f(U) \), where \( f(MW) \) is a closed normal analytic subgroup of \( L \) and \( f(U) \) is a toral group. Moreover, \( f(MW) \cap f(U) \) is finite, and \( f(W) \cap (f(M) \cdot f(U)) = f(W) \cap Z(L) = \{e\} \).

(iii) If \( L \) is (CA) and \( Z(L) \) is compact, then \( W = \{e\} \).

**Proof.** Since \( Z(G) \) is compact, we can conclude from van Est [2] that \( G \) is non-(CA). We will now maintain the notation of the main structure theorem, as well as the notation in the proof of the above lemma.

Since \( Z(G) \) is compact, we know from the above lemma that there exists a maximal analytic subgroup \( J \) of \( G \) which contains \( M \), such that \( f(J) \) is closed in \( L \). Then from Goto [1] there exists a closed vector subgroup \( U \) of \( G \) such that \( G = JU \), \( J \cap U = \{e\} \). Moreover, \( L = f(J) \cdot f(U) \), where \( f(U) \) is a toral group and \( f(J) \cap f(U) \) is finite.

In the proof of Goto's theorem [1], applied to \( I(G) \), \( T' \) is a closed central subgroup of an arbitrarily fixed maximal compact subgroup \( K \) of \( \overline{I(G)} \). We will assume that \( K \) has been selected so that it contains \( pGL(f(U)) \).

Now let \( \pi: G \rightarrow V \) be the natural projection and let \( W = \pi(J) \). Then \( W \) is a closed vector subgroup of \( V \) and since \( J \) contains \( M \) we see that \( J = M \cdot W \), \( M \cap W = \{e\} \). Therefore,

\[
L = f(J) \cdot f(U) = f(M) \cdot f(W) = f(U) \cdot f(U) \cdot f(W),
\]

where \( f(M) \cdot f(U) \) is a closed normal analytic subgroup of \( L \). Since \( f(U) \cap f(J) \) is finite and contained in \( f(G) \), it is contained in \( f(M) \). Hence, if \( (f(M) \cdot f(U)) \cap f(W) \neq \{e\} \), then \( f(w) = f(m) \cdot x, x \in f(U) \), and so \( x = f(m)^{-1} \cdot f(w) \). Hence, \( x \in f(U) \cap f(J) \), which is contained in \( f(M) \). By the uniqueness of the decomposition in \( J \), we have \( w = e \). So \( L = (f(M) \cdot f(U)) \cdot f(W) \), \( (f(M) \cdot f(U)) \cap f(W) = \{e\} \). Moreover, \( f(W) \cap Z(L) = \{e\} \), since \( f(J) \cap Z(L) \) is contained in \( f(M) \), and \( W \cap M = \{e\} \). Hence, \( L = (f(M) \cdot f(U)) \circ \varphi(W) \).

We now show that \( UW \) is a closed vector subgroup of \( G = MWU \). Let \( \varphi: W \rightarrow A(MU) \) be given by \( \varphi(w)(mu) = w(mu)w^{-1} \). Since \( W \cap Z(G) = \{e\} \), \( \varphi \) is an immersion and so \( G = MU \circ W \). Since the image of each one-parameter subgroup of \( W \) under \( I_G \) is not closed in \( A(G) \), we see from Zerling [4] that \( \overline{\varphi(W)} \) is a toral group.

Let \( u \in U \). Then \( I_G(\varphi(w) \cdot u) = I_G(wuw^{-1}) = I_G(u) \), since \( I_G(w) \in T' \) commutes with \( I_G(u) \in \rho_{GL}(f(U)) \) because of our selection of \( K \) above. Hence \( \sigma(u) \cdot u^{-1} \in Z(G) \) for all \( \sigma \in \overline{\varphi(W)} \). Since \( Z(G) \) is a closed central subgroup of \( MU \) and each element of \( \overline{\varphi(W)} \) keeps \( Z(G) \) elementwise fixed, we see from Zerling [3, Lemma 2.2] that \( \sigma(u) = u \) for all \( \sigma \in \overline{\varphi(W)} \). Therefore, \( UW \) is a closed vector subgroup of \( G \).

Now suppose that \( L \) is (CA) and \( Z(L) \) is compact. As we did above, we can show that the closure of \( f(W) \) in \( A(f(M) \cdot f(U)) \) is a toral group, call it \( T_3 \). Then \( L \) is properly dense in \( f(M) \cdot f(U) \) \( \circ T_3 \). This is a contradiction from van Est [2]. Hence \( W = \{e\} \). This completes the proof of our theorem.
3. An example. The following example shows that in the above theorem \( W \) need not be \( \{e\} \), even if \( L \) is \((CA)\). Let \( G = MV \) be any non-\((CA)\) analytic group and suppose that the dimension of \( V \) is at least two. Such a group \( G \) can easily be obtained by a slight modification of the example in Zerling [3].

We continue with our previous notation and let \( S = G \otimes T' \). Since \( V \) is dense in \( T \) we can select an element \( v_0 \in V \) such that \( v_0 \) generates a dense subgroup of \( T \). Then \( v'_0 = I_G(v_0) \) generates a dense subgroup of \( T' \). Let \( D \) denote the subgroup of \( S \) generated by \( (v_0, v'_0^{-1}) \). Then \( D \) is a free discrete central subgroup of \( S \) and \( L = S/D \) is a \((CA)\) analytic group for which \( g \mapsto (g, e)D \) is a dense immersion \( f \) of \( G \) into \( L \); see Zerling [4, the proof of Theorem 2.2].

Next let \( V_\lambda \) be the one-dimensional vector subgroup of \( V \) containing \( v_0 \) and let \( V_\mu \) be a vector subgroup of \( V \) such that \( V = V_\lambda \cdot V_\mu, V_\lambda \cap V_\mu = \{e\} \). Let \( \delta: S \to L \) be the canonical projection. Then \( G = MV_\lambda, f(MV_\mu) \) is closed in \( L \) and \( f(V_\lambda) = f(V_\lambda) \cdot \delta(T') \) is a toral group. \( L = f(MV_\mu) \cdot f(V_\lambda) \), and \( f(MV_\mu) \cap f(V_\lambda) \) is trivial. Hence \( V_\mu = W \neq \{e\} \).

Bibliography


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