INDECOMPOSABLE COMPACT PERTURBATIONS OF THE
BILATERAL SHIFT

DOMINGO A. HERRERO

Abstract. Recent results of M. Radjabalipour and H. Radjavi assert that
the sum of a normal operator N with spectrum on a smooth Jordan curve
and a compact operator K in the Macaev ideal $\mathfrak{S}_{\omega}$ is decomposable
provided the spectrum of $N + K$ does not fill the interior of the curve.
Examples are given to show that this result cannot be essentially improved
by taking $K$ in a larger ideal.

1. Let $L(\mathcal{H})$ be the algebra of all (bounded linear) operators on a complex
Hilbert space $\mathcal{H}$. The Macaev ideal $\mathfrak{S}_{\omega}$ is the set of all compact operators $K$
in $L(\mathcal{H})$ such that $\sum_{n=1}^{\infty} \mu(n)/n < \infty$, where $\mu(1), \mu(2), \ldots, \mu$, are the eigenvalues of $(K*K)^{1/2}$ arranged in decreasing order and repeated according to
multiplicity.

Compact perturbations of normal operators with spectrum on a smooth
Jordan curve by an operator $K \in \mathfrak{S}_{\omega}$ are known to have a rich family of
invariant subspaces. To make this precise, several definitions will be neces-
sary:

An invariant subspace $\mathcal{M}$ of $T \in L(\mathcal{H})$ is a maximal spectral subspace of
$T$ if $\mathcal{M} \subset \mathcal{M}$ for all invariant subspaces $\mathcal{M}$ of $T$ such that the spectrum
$\Lambda(T|\mathcal{M})$ of the restriction of $T$ to $\mathcal{M}$ is contained in $\Lambda(T|\mathcal{M})$. $T$ is
decomposable (in the sense of [1]) if for every finite covering $G_j$, $j = 1, 2, \ldots, n$, of $\Lambda(T)$ there exists a set of maximal spectral subspaces $\mathcal{K}_j$, $j = 1, 2, \ldots, n$, of $T$ such that $\Lambda(T|\mathcal{K}_j) \subset G_j$, $j = 1, 2, \ldots, n$, and $\mathcal{H} = \mathcal{K}_1 + \mathcal{K}_2 + \cdots + \mathcal{K}_n$. Moreover, $T$ is called strongly decomposable if $T|\mathcal{M}$ is decom-
posable for every maximal spectral subspace $\mathcal{M}$.

M. Radjabalipour and H. Radjavi [10]–[13] have improved a result of V. I.
Macaev about compact perturbations of hermitian operators [9] by proving
the following:

(i) Let $T$ be the sum of an operator $A$ having spectrum on a $C^2$ Jordan
curve $J$ and a compact operator $K \in \mathfrak{S}_{\omega}$. Assume that $\|(z - A)^{-1}\| \leq C/d(z)$, where $d(z)$ is the distance from $z$ to $\Lambda(A)$ and $C > 1$ is a constant

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independent of $z$, and that $\Lambda(T)$ does not fill the interior of $J$. Then $T$ is strongly decomposable.

(ii) Assume that $\Lambda(T)$ is contained in a $C^2$ Jordan curve $J$ and that there exist a positive number $\varepsilon$ and a nonincreasing function $M: (0, \varepsilon) \to (0, \infty)$ such that $\int_0^\varepsilon \log^2 M(t) \, dt < \infty$ (log$^m$ $x$ denotes the $m$th iterated logarithm). If $\|z - T\|^{-1} \leq M[d(z)]$ for $z \notin J$, then $T$ is strongly decomposable. This is true, in particular, if $M(t) = \exp(\exp(t^{-p}))$, $0 < p < 1$.

It will be shown that these two results are essentially sharp by proving that compact perturbations of the (unitary!) bilateral shift $U$ in $l^2$ (defined by $Ue_n = e_{n+1}$) fail to be decomposable.

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2. Let $\{a_n\}$ be a bounded two-sided sequence of positive reals and define the bilateral weighted shift $B$ in $l^2$ by $Be_n = a_n e_{n+1}$, $n \in \mathbb{Z}$. If \( \lim_{n \to \pm \infty} a_n = 1 \), then $B - U$ is compact and $\Lambda(B)$ is the boundary $\partial D$ of the unit disc $D = \{z: |z| < 1\}$.

Define $w_0 = 1$, $w_n = a_0, a_1 \cdots a_{n-1}$ and $w_{-n} = (a_{n-1} a_{n-2} \cdots a_0)^{-1}$ for all positive $n$. Then $B$ is unitarily equivalent to multiplication by $z$ on the space

$$L^2(w = (w_n)) = \left\{ f(z) = \sum_{n \in \mathbb{Z}} b_n z^n: \|f\|^2 = \sum |b_n w_n|^2 < \infty \right\}$$

of formal Laurent series [3], [6]. Now we are in a position to construct the counterexamples.

**Theorem.** Let $\varphi = \{\varphi_n\}_1^\infty$ be a nonincreasing sequence of positive reals and assume that: (i) $\varphi_n \log en$ is nonincreasing, (ii) $n(\varphi_n - \varphi_{n+1}) \to 0$, and (iii) $\Sigma \varphi_n / n = \infty$. Let $B$ be the bilateral weighted shift defined by $Be_n = a_n e_{n+1}$, where $a_n = w_{n+1} / w_n$, $w_0 = 1$ and $w_n = w_{-n} = \exp(n \varphi_n)$ for $n = 1, 2, \ldots$.

Then

(a) Every nonzero invariant subspace $\mathcal{M}$ of $B$ is either invariant under $B^{-1}$ and satisfies $\Lambda(B|\mathcal{M}) = \partial D = \Lambda(B)$, or it is not invariant under $B^{-1}$ and $\Lambda(B|\mathcal{M}) = D$ is the closed unit disc.

(b) $K = B - U$ is a compact operator such that the eigenvalues of $(K^*K)^{1/2}$ (ordered as above indicated) satisfy $\mu_{2n-1}, \mu_{2n} \leq C \varphi_n$, where $C$ is a constant independent of $n$.

(c) $\log^2 \|z - B\|^{-1} \leq M / d(z)$, for a suitable constant $M$.

**Proof.** (a) This will follow by using the same kind of “sandwich theory” as in [4]:

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Let \( L^1(w) = \{ f(z) = \sum b_n z^n : \| f \|_1 + \| b_n w_n \| < \infty \} \) and let \( L^1(\sqrt{w}) \) be similarly defined, with \( w_n \) replaced by \( \sqrt{w_n} \), \( n \in \mathbb{Z} \) (the norm of an element of \( L^1(\sqrt{w}) \) will be denoted by \( \| \cdot \|'' \)). Then, for every finite sum \( f(z) = \sum b_n w_n \), by the Cauchy-Schwarz inequality we have

\[
\| f \|'' = \sum |b_n \sqrt{w_n}| = \sum |b_n w_n|/\sqrt{w_n}
\leq \left( \sum |b_n w_n|^2 \right)^{1/2} \left( \sum 1/w_n \right)^{1/2} = C \| f \|,
\]

where \( C = (\sum 1/w_n)^{1/2} < \infty \) is a constant independent of \( f \). It readily follows that \( (1/C) \| f \|'' < \| f \| < \| f \|_1 \) and, therefore, that \( L^1(w) \subset L^2(w) \subset L^1(\sqrt{w}) \) and both inclusions are continuous.

It follows from (i) and (ii) that \( \lim_{n \to \pm \infty} a_n = 1 \) and, therefore, \( B \) is a compact perturbation of \( U \) and \( \Lambda(B) = \partial D \) (see [3], [4], [6]). Moreover, it follows from (i)--(iii) and the results of [2], [4] that \( L^1(w) \) and \( L^1(\sqrt{w}) \) are actually algebras (under pointwise multiplication) of quasi-analytic functions defined on \( \partial D \). I.e., \( L^1(\sqrt{w}) \) is contained in \( C^\infty(\partial D) \); if \( f, g \in L^1(w) \) \( (L^1(\sqrt{w}), \) resp.) and the vanishing of \( f(z) \in L^1(\sqrt{w}) \) together with all its derivatives at some point \( \lambda \in \partial D \) implies that \( b_n = 0 \) for all \( n \).

Let \( \mathcal{M} \) be a nonzero invariant subspace of \( B \). If \( \mathcal{M} \) is not invariant under \( B^{-1} \), then \( 0 \in \Lambda(B| \mathcal{M}) \) and it follows that \( \Lambda(B| \mathcal{M}) = D^- \); if \( \mathcal{M} \) is also invariant under \( B^{-1} \), then \( \Lambda(B| \mathcal{M}) \subset \partial D \) (these two results easily follow from [7], [8]).

In the second case, the closure \( \mathfrak{R} \) of \( \mathcal{M} \) in \( L^1(\sqrt{w}) \) is an ideal of this algebra and we can proceed as in [4] in order to show that

\[
\mathfrak{R} = \bigcap_{j=1}^m \{ f \in L^1(\sqrt{w}) : f(z_j) = f'(z_j) = \ldots = f^{(m_j-1)}(z_j) = 0 \}
\]

for a finite set of points \( z_j \in \partial D, j = 1, 2, \ldots, m \), and positive integers \( m_j \), \( j = 1, 2, \ldots, m \), so that \( \dim L^1(\sqrt{w})/\mathfrak{R} = n = \sum m_j m_j < \infty \). Clearly, \( \mathfrak{R} \subset \mathcal{M} \cap L^2(w) \) and for every \( \lambda \in \partial D \setminus \{ z_1, z_2, \ldots, z_m \} \) there exists a function \( h_\lambda \in \mathfrak{R} \) such that \( h_\lambda(\lambda) \neq 0 \).

Let \( \mathfrak{R}_0 (\mathfrak{R}'' \) be the closure of \( (B - \lambda) \mathfrak{R} \) in \( L^2(w) \) (in \( L^1(\sqrt{w}) \), resp.). Since \( (B - \lambda)f(z) = (z - \lambda)f(z) \), it readily follows that \( \mathfrak{R}'' = \mathfrak{R} \cap \{ f \in L^1(\sqrt{w}) : f(\lambda) = 0 \} \) has codimension 1 in \( \mathfrak{R} \). In particular, \( h_\lambda \in \mathfrak{R} \setminus \mathfrak{R}'' \), a fortiori, \( h_\lambda \not\subset \mathfrak{R}_0 (\mathfrak{R}_0 \subset \mathfrak{R} \cap \mathfrak{R}'' \) and, therefore, \( (B - \lambda) \mathcal{M} \) is not dense in \( \mathfrak{R} \). Hence, \( \lambda \in \Lambda(B| \mathcal{M}) \).

Since this holds for all but finitely many \( \lambda \)'s in \( \partial D \), it is not difficult to conclude that \( \partial D \subset \Lambda(B| \mathcal{M}) \) and, therefore, \( \Lambda(B| \mathcal{M}) = \partial D \), as promised.

If \( \mathcal{M} \) contains a nonzero function \( L^1(w) \), it is easy to see that \( \mathcal{M} \cap L^1(w) \) is actually an ideal of finite codimension \( n \) in \( L^1(w) \) and, a fortiori, that \( \mathcal{M} \) also has codimension \( n \) in \( L^2(w) \). (The details are left to the reader. The author conjectured that \( \mathcal{M} \cap L^1(w) \neq \{0\} \) for every invariant subspace \( \mathcal{M} \), but was unable to prove it.)
(b) This follows immediately from $K e_n = (B - U)e_n = (a_n - 1)e_{n+1}$ (see [5]).

(c) Since $U$ is normal, $\gamma(t) = \max \{ \| (z - U)^{-1} \| : d(z) = t \} = 1/t$. On the other hand (b) and condition (i) show that $\mu_n < k/\log en$, for some constant $k > 0$ and for all $n = 1, 2, \ldots$, and, therefore, $\nu(t) = \max \{ n : \mu_n > 1/t \} < e^{kt}$. Now we are in a position to apply Theorem 1 of [13], whence the result follows.

\[ \square \]

3. The sequences

$$\left\{ \varphi_n = \varphi_n(m) = \left[ \log M_1 n \cdot \log(2) M_2 n \cdot \ldots \cdot \log(m) M_m n \right]^{-1} \right\},$$

where $M_1 = e$ and $M_{k+1} = \exp(M_k)$ for $k = 1, 2, \ldots, m-1$ ($m = 1, 2, \ldots$), are concrete examples of sequences satisfying conditions (i)-(iii). Moreover, if $m_n \to \infty$ and

$$\varphi_n = \psi_n(m_n) = \varphi_n(m_n)/\left[ \log(m_n+1) M_{m_n+1} n \right]^2$$

satisfies (iii), then it actually satisfies (i)-(iii).

Given an arbitrary increasing function $h(n)$ such that $h(1) = 1$ and $\lim_{n \to \infty} h(n) = \infty$, it is not difficult to find a sequence $\{m_n\}_{n=1}^{\infty}$ such that $\{\psi_n(m_n)\}$ satisfies (i)-(iii), but $\sum \psi_n(m_n)/[nh(n)] < \infty$. Now we can give a very precise meaning to our introductory sentence "these two results are essentially sharp":

**Corollary.** Let $h(n)$ be an arbitrary increasing function such that $h(1) = 1$ and $\lim_{n \to \infty} h(n) = \infty$, and let $\mathcal{E}_h = \{ K \in \mathcal{E}(F) : K$ is compact and $\sum \mu(n)/[nh(n)] < \infty \}$. Then there exists an operator $K$ in the ideal $\mathcal{E}_h$ such that $U + K$ is indecomposable, even when its resolvent satisfies the growth conditions of the Theorem.

The proof follows immediately from the Theorem and the above observations.

4. **An example.** Let $\varphi_n = (\log en \cdot \log(2)e^2 n)^{-1}$; then a straightforward computation shows that $\mu_n$ is of the same order of magnitude as $\varphi_n$ ($n = 1, 2, \ldots$) and that, in this case, $\nu(t) < \exp(kt/\log t)$, whence (by applying once again Theorem 1 of [13]) we can improve the estimation (c) of the Theorem to

$$\log(2) \| (z - B)^{-1} \| < M/\left[ d(z) \cdot \log 1/d(z) \right],$$

for a suitable constant $M$ and for $0 < d(z) < \frac{1}{2}$.

**References**


DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD SIMÓN BOLÍVAR, CARACAS, VENEZUELA

Current address: Departamento de Matemáticas, Instituto Venezolano de Investigaciones Científicas, Caracas, Venezuela