INDECOMPOSABLE COMPACT PERTURBATIONS OF THE
BILATERAL SHIFT

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Abstract. Recent results of M. Radjabalipour and H. Radjavi assert that the sum of a normal operator \( N \) with spectrum on a smooth Jordan curve and a compact operator \( K \) in the Macaev ideal \( \mathcal{E}_\omega \) is decomposable provided the spectrum of \( N + K \) does not fill the interior of the curve. Examples are given to show that this result cannot be essentially improved by taking \( K \) in a larger ideal.

1. Let \( \mathcal{L}(\mathcal{H}) \) be the algebra of all (bounded linear) operators on a complex Hilbert space \( \mathcal{H} \). The Macaev ideal \( \mathcal{E}_\omega \) is the set of all compact operators \( K \) in \( \mathcal{L}(\mathcal{H}) \) such that \( \sum n \mu(n)/n < \infty \), where \( \mu(1), \mu(2), \ldots \), are the eigenvalues of \( (K^*K)^{1/2} \) arranged in decreasing order and repeated according to multiplicity.

Compact perturbations of normal operators with spectrum on a smooth Jordan curve by an operator \( K \in \mathcal{E}_\omega \) are known to have a rich family of invariant subspaces. To make this precise, several definitions will be necessary:

An invariant subspace \( \mathcal{M} \) of \( T \in \mathcal{L}(\mathcal{H}) \) is a maximal spectral subspace of \( T \) if \( \mathcal{M} \subset \mathcal{M} \) for all invariant subspaces \( \mathcal{M} \) of \( T \) such that the spectrum \( \Lambda(T|\mathcal{M}) \) of the restriction of \( T \) to \( \mathcal{M} \) is contained in \( \Lambda(T|\mathcal{M}) \). \( T \) is decomposable (in the sense of [1]) if for every finite covering \( G_j, j = 1, 2, \ldots, n \), of \( \Lambda(T) \) there exists a set of maximal spectral subspaces \( \mathcal{M}_j, j = 1, 2, \ldots, n, \) of \( T \) such that \( \Lambda(T|\mathcal{M}_j) \subset G_j, j = 1, 2, \ldots, n, \) and \( \mathcal{H} = \mathcal{M}_1 + \mathcal{M}_2 + \cdots + \mathcal{M}_n \). Moreover, \( T \) is called strongly decomposable if \( T|\mathcal{M} \) is decomposable for every maximal spectral subspace \( \mathcal{M} \).

M. Radjabalipour and H. Radjavi [10]–[13] have improved a result of V. I. Macaev about compact perturbations of hermitian operators [9] by proving the following:

(i) Let \( T \) be the sum of an operator \( A \) having spectrum on a \( C^2 \) Jordan curve \( J \) and a compact operator \( K \in \mathcal{E}_\omega \). Assume that \( ||(z - A)^{-1}|| < C/d(z) \), where \( d(z) \) is the distance from \( z \) to \( \Lambda(A) \) and \( C > 1 \) is a constant.

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independent of \( z \), and that \( \Lambda(T) \) does not fill the interior of \( J \). Then \( T \) is strongly decomposable.

(ii) Assume that \( \Lambda(T) \) is contained in a \( C^2 \) Jordan curve \( J \) and that there exist a positive number \( \varepsilon \) and a nonincreasing function \( M: (0, \varepsilon) \rightarrow (0, \infty) \) such that \( \int_0^1 \log^2 M(t) \, dt < \infty \) (\( \log^m x \) denotes the \( m \)th iterated logarithm). If \( ||(z - T)^{-1}|| < M[d(z)] \) for \( z \not\in J \), then \( T \) is strongly decomposable. This is true, in particular, if \( M(t) = \exp(\exp t^{-p}) \), \( 0 < p < 1 \).

It will be shown that these two results are essentially sharp by proving that compact perturbations of the (unitary!) bilateral shift \( U \) in \( \ell^2 \) (defined by \( Ue_n = e_{n+1} \) for all \( n \) in the set \( \mathbb{Z} \) of all integers, where \( \{e_n\} \) is the canonical basis of \( \ell^2 \)) by an operator in a certain class of ideals which are only "slightly larger" than \( \mathfrak{S}_\omega \) fail to be decomposable.

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2. Let \( \{a_n\} \) be a bounded two-sided sequence of positive reals and define the bilateral weighted shift \( B \) in \( \ell^2 \) by \( Be_n = a_ne_{n+1}, n \in \mathbb{Z} \). If \( \lim_{n \to \pm \infty} a_n = 1 \), then \( B - U \) is compact and \( \Lambda(B) \) is the boundary \( \partial D \) of the unit disc \( D = \{z: |z| < 1\} \).

Define \( w_0 = 1, w_n = a_0, a_1 \cdots a_{n-1} \) and \( w_{-n} = (a_1 a_2 \cdots a_{-n})^{-1} \) for all positive \( n \). Then \( B \) is unitarily equivalent to multiplication by \( z \) on the space

\[
L^2(w = (w_n)) = \left\{ f(z) = \sum_{n \in \mathbb{Z}} b_n z^n: \|f\|^2 = \sum |b_n w_n|^2 < \infty \right\}
\]

of formal Laurent series [3], [6]. Now we are in a position to construct the counterexamples.

**Theorem.** Let \( \varphi = \{\varphi_n\}_{n=1}^\infty \) be a nonincreasing sequence of positive reals and assume that: (i) \( \{\varphi_n \log n\} \) is nonincreasing, (ii) \( n(\varphi_n - \varphi_{n+1}) \to 0 \), and (iii) \( \sum \varphi_n / n = \infty \). Let \( B \) be the bilateral weighted shift defined by \( Be_n = a_ne_{n+1} \), where \( a_n = w_{n+1} / w_n \), \( w_0 = 1 \) and \( w_n = \exp(n\varphi_n) \) for \( n = 1, 2, \ldots \).

Then

(a) Every nonzero invariant subspace \( \mathcal{M} \) of \( B \) is either invariant under \( B^{-1} \) and satisfies \( \Lambda(B|\mathcal{M}) = \partial D = \Lambda(B) \), or it is not invariant under \( B^{-1} \) and \( \Lambda(B|\mathcal{M}) = D^- \) is the closed unit disc.

(b) \( K = B - U \) is a compact operator such that the eigenvalues of \( (K^*K)^{1/2} \) (ordered as above indicated) satisfy \( \mu_{2n-1} \leq C\varphi_n \), where \( C \) is a constant independent of \( n \).

(c) \( \log^2 ||(z - B)^{-1}|| < M/d(z) \), for a suitable constant \( M \).

**Proof.** (a) This will follow by using the same kind of "sandwich theory" as in [4]:

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Let $L'(w) = \{ f(z) = \sum b_n z^n : \| f \|_1 + \sum |b_n w_n| < \infty \}$ and let $L'(\sqrt{w})$ be similarly defined, with $w_n$ replaced by $\sqrt{w_n}$, $n \in \mathbb{Z}$ (the norm of an element of $L'(\sqrt{w})$ will be denoted by $\| \cdot \|''$). Then, for every finite sum $f(z) = \sum b_n w_n$, by the Cauchy-Schwarz inequality we have

$$\|f\|'' = \sum |b_n \sqrt{w_n}| = \sum |b_n w_n| / \sqrt{w_n} \leq \left( \sum |b_n w_n|^2 \right)^{1/2} \left( \sum 1 / w_n \right)^{1/2} = C \|f\|,$$

where $C = (\sum 1 / w_n)^{1/2} < \infty$ is a constant independent of $f$. It readily follows that $(1/C)\|f\|'' \leq \|f\| \leq \|f\|_1$ and, therefore, that $L'(w) \subset L^2(w) \subset L'(\sqrt{w})$ and both inclusions are continuous.

It follows from (i) and (ii) that $\lim_{n \to \pm \infty} a_n = 1$ and, therefore, $B$ is a compact perturbation of $U$ and $\Lambda(B) = \partial D$ (see [3], [4], [6]). Moreover, it follows from (i)-(iii) and the results of [2], [4] that $L'(w)$ and $L'(\sqrt{w})$ are actually algebras (under pointwise multiplication) of quasi-analytic functions defined on $\partial D$. I.e., $L'(\sqrt{w})$ is contained in $C^\infty(\partial D)$; if $f, g \in L'(w)$ ($L'(\sqrt{w})$, resp.) and the vanishing of $f(z) \in L'(\sqrt{w})$ together with all its derivatives at some point $\lambda \in \partial D$ implies that $b_n = 0$ for all $n$.

Let $\mathcal{M}$ be a nonzero invariant subspace of $B$. If $\mathcal{M}$ is not invariant under $B^{-1}$, then $0 \in \Lambda(B|\mathcal{M})$ and it follows that $\Lambda(B|\mathcal{M}) = D$; if $\mathcal{M}$ is also invariant under $B^{-1}$, then $\Lambda(B|\mathcal{M}) \subset \partial D$ (these two results easily follow from [7], [8]).

In the second case, the closure $\mathcal{M}$ of $\mathcal{M}$ in $L'(\sqrt{w})$ is an ideal of this algebra and we can proceed as in [4] in order to show that

$$\mathcal{M} = \bigcap_{j=1}^{m} \{ f \in L'(\sqrt{w}) : f(z_j) = f'(z_j) = \cdots = f^{(m_j-1)}(z_j) = 0 \}$$

for a finite set of points $z_j \in \partial D$, $j = 1, 2, \ldots, m$, and positive integers $m_j$, $j = 1, 2, \ldots, m$, so that $\dim L'(\sqrt{w})/\mathcal{M} = n = \sum_{j=1}^{m} m_j < \infty$. Clearly, $\mathcal{M} \subset \mathcal{M} \subset L^2(w)$ and for every $\lambda \in \partial D \setminus \{ z_1, z_2, \ldots, z_m \}$ there exists a function $h_\lambda \in \mathcal{M}$ such that $h_\lambda(\lambda) \neq 0$.

Let $\mathcal{M}_0 (\mathcal{M}''')$ be the closure of $(B - \lambda)\mathcal{M}$ in $L^2(w)$ (in $L'(\sqrt{w})$, resp.). Since $(B - \lambda)f(z) = (z - \lambda)f(z)$, it readily follows that $\mathcal{M}'' = \mathcal{M} \cap \{ f \in L'(\sqrt{w}) : f(\lambda) = 0 \}$ has codimension 1 in $\mathcal{M}$. In particular, $h_\lambda \in \mathcal{M} \setminus \mathcal{M}''$, a fortiori, $h_\lambda \notin \mathcal{M}_0 (\mathcal{M}''')$ and, therefore, $(B - \lambda)\mathcal{M}$ is not dense in $\mathcal{M}$. Hence, $\lambda \in \Lambda(B|\mathcal{M})$.

Since this holds for all but finitely many $\lambda$'s in $\partial D$, it is not difficult to conclude that $\partial D \subset \Lambda(B|\mathcal{M})$ and, therefore, $\Lambda(B|\mathcal{M}) = \partial D$, as promised.

If $\mathcal{M}$ contains a nonzero function $L'(w)$, it is easy to see that $\mathcal{M} \cap L'(w)$ is actually an ideal of finite codimension $n$ in $L'(w)$ and, a fortiori, that $\mathcal{M}$ also has codimension $n$ in $L^2(w)$. (The details are left to the reader. The author conjectured that $\mathcal{M} \cap L'(w) \neq \{0\}$ for every invariant subspace $\mathcal{M}$, but was unable to prove it.)
(b) This follows immediately from $K_{En} = (B - U)e_n = (a_n - 1)e_{n+1}$ (see [5]).

(c) Since $U$ is normal, $\gamma(t) = \max\{\|z - U\|^{-1}\}: d(z) = t\} = 1/t$. On the other hand (b) and condition (i) show that $\mu_n < k/\log en$, for some constant $k > 0$ and for all $n = 1, 2, \ldots$, and, therefore, $\nu(t) = \max\{n: \mu_n > 1/t\} < e^{kt}$. Now we are in a position to apply Theorem 1 of [13], whence the result follows. □

3. The sequences

$$\left\{ q_n = q_n(m) = \left[ \log M_1 n \cdot \log^{(2)} M_2 n \cdots \log^{(m)} M_m n \right]^{-1} \right\},$$

where $M_1 = e$ and $M_{k+1} = \exp(M_k)$ for $k = 1, 2, \ldots, m - 1$ ($m = 1, 2, \ldots$), are concrete examples of sequences satisfying conditions (i)–(iii).

Moreover, if $m_n \to \infty$ and

$$q_n = q_n(m_n) = q_n(m_n)/\left[ \log^{(m+1)} M_{m+1} n \right]^2$$

satisfies (iii), then it actually satisfies (i)–(iii).

Given an arbitrary increasing function $h(n)$ such that $h(1) = 1$ and $\lim_{n \to \infty} h(n) = \infty$, it is not difficult to find a sequence $\{m_n\}_{n=1}^{\infty}$ such that $\{q_n(m_n)\}$ satisfies (i)–(iii), but $\sum q_n(m_n)/[nh(n)] < \infty$. Now we can give a very precise meaning to our introductory sentence “these two results are essentially sharp”:

**Corollary.** Let $h(n)$ be an arbitrary increasing function such that $h(1) = 1$ and $\lim_{n \to \infty} h(n) = \infty$, and let $\mathcal{E}_h = \{K \in \mathcal{E}(\hat{F}): K$ is compact and $\sum \mu(n)/[nh(n)] < \infty\}$. Then there exists an operator $K$ in the ideal $\mathcal{E}_h$ such that $U + K$ is indecomposable, even when its resolvent satisfies the growth conditions of the Theorem.

The proof follows immediately from the Theorem and the above observations.

4. An example. Let $q_n = (\log^2 e \cdot e^n)^{-1}$; then a straightforward computation shows that $\mu_n$ is of the same order of magnitude as $q_n$ ($n = 1, 2, \ldots$) and that, in this case, $\nu(t) < \exp(kt/\log t)$, whence (by applying once again Theorem 1 of [13]) we can improve the estimation (c) of the Theorem to

$$\log^{(2)}\|z - B\|^{-1} \leq M/\left[ d(z) \cdot \log 1/d(z) \right],$$

for a suitable constant $M$ and for $0 < d(z) < \frac{1}{2}$.

**References**


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